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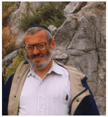
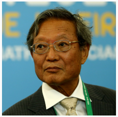
    

Adrien Brochier

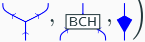
WKO in caen (2012)

- This talk will be an overview of some diagrammatic/universal constructions related to Etingof–Kazhdan's quantization of Lie bialgebras.
- An important feature of that construction is that it's functorial/acyclic, but I'd like to emphasize special features that arise when you also allow cycles (i.e. duality and traces).
- Inspired by work of Alekseev–Torossian and Bar-Natan–Dancso I'll explain how a generalization of the Duflo isomorphism and of (a version of) the Kashiwara–Vergne conjecture pop up in this framework. . .
- . . . in a way that I'd argue is somewhat close in spirit to KV's original approach.
- Warning: there will be lots of " ".

- Let $\mathbb{Q} \subset \mathbf{k}$ be a commutative ring and let[1]

$$\Big(\mathfrak{g},$$  $$\Big)$$

  be a Lie bialgebra over $\mathbf{k}$.
- This induces on $G = \exp(\mathfrak{g})$ the structure of a Poisson group.
- This makes

$$\mathcal{O}(G) := \Big(S(\mathfrak{g}^*),$$ 

$$\Big),$$

  where 

is induced by the Baker–Campbell–Hausdorff formula, into a Poisson-Hopf algebra.

Theorem (Etingof–Kazhdan, see also Ševera–Pulmann)

For any choice of a Drinfeld associator Φ, there exists a Hopf algebra

$$\mathcal{O}_\Phi(G) := (S(\mathfrak{g}^*), m_\Phi, \Delta_\Phi, S_\Phi)$$

which deform the Poisson bracket on G and such that

$$\Delta_\Phi =$$ 

$$+ \dots .$$

Moreoever this is functorial, given by a morphim of PROPs (aka “acyclic formulas”).

[1] I assume everything’s (co)nilpotent

- If $\mathfrak{g}$ is finite-dimensional, then

$$\mathfrak{g}^\vee := \Big(\mathfrak{g}^*,$$  $$, -$$  $$\Big)$$

  is also a Lie bialgebra.

- G and $G^\vee$ Poisson act on each other via their respective Poisson brackets (the "dressing actions").

"Duflo theorem"

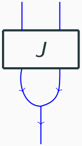
If $\mathfrak{g}$ is finite-dimensional, then there exists a diff. operator u_Φ on $G^\vee$ such that

$$u_\Phi : \mathcal{O}(G^\vee)^G \hookrightarrow \mathcal{O}_\Phi(G^\vee)$$

is an algebra morphism.

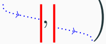
- If  $= 0$, then
  - $G^\vee = (\mathfrak{g}^*, +)$ with the KKS bracket,
  - $\mathcal{O}_\Phi(G^\vee) \cong U(\mathfrak{g})$ canonically,
  - the dressing action is the adjoint action of $\mathfrak{g}$ on $S(\mathfrak{g})$.
- This recovers the original Duflo theorem: $U(\mathfrak{g})^\mathfrak{g} \cong S(\mathfrak{g})^\mathfrak{g}$ as algebras.

- Proof “à la Kashiwara–Vergne”:
    1. Write the product on $\mathcal{O}_\Phi(G^\vee)$ as

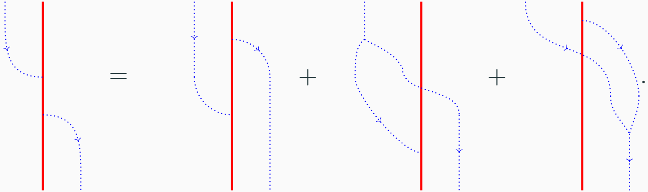


       where J is a diff. operator on $G^\vee \times G^\vee$ of a particular kind.
    2. Compute the action of J on $\mathcal{O}(G^\vee)^G \otimes \mathcal{O}(G^\vee)^G$ and show this can be trivialized.
    3. KV’s strategy is based on a clever Ansatz for the meaning of “particular kind” in the case of Lie algebras (it should be the exponential of a “tangential derivation”).
    4. For those the failure to be trivial on invariants is measured by a certain divergence cocycle, so this reduces to a cohomological problem.
    5. Alekseev–Torossian connected this to a seemingly different non-commutative cohomological problem, the so-called “twist equation”.
    6. The twist equation is modelled on the axioms for monoidal functors, and a version of that equation is the key ingredient in EK theory.
- This talk: explain that in the general case, any solution of the twist equation is gauge equivalent to one that acts trivially on invariants.

- Let YD be the symmetric monoidal category (PROP) freely generated by a Lie bialgebra and a Yetter–Drinfeld module

$\left(V, \right.$  $\left. \right).$

- Recall it means V is both a $\mathfrak{g}$-module and a $\mathfrak{g}$-comodule, and



- $\mathrm{End}(V^{\otimes n})$ is the algebra of acyclic arrow/virtual Jacobi diagrams.
- "Normal ordering" where all coactions appear before actions (but not in general the other way around!)
- Can also be described in terms of free algebras (Enriquez, Appel-Toldeano–Laredo, Rivezzi).

- Let $\mathcal{YD}_{\mathfrak{g}}$ by the symmetric monoidal category of all YD-modules over $\mathfrak{g}$, and let $F : \mathcal{YD}_{\mathfrak{g}} \to \mathrm{YD}$ the forgetful functor.
- Let $\underline{\mathcal{O}}(G) \in \mathcal{YD}_{\mathfrak{g}}$ be $\mathcal{O}(G)$ equipped with the action of $\mathfrak{g}$ on G via invariant vector fields and the dressing coaction.
- Equipped with its original product this is a commutative algebra in $\mathcal{YD}_{\mathfrak{g}}$.

Fundamental theorem of Hopf algebras

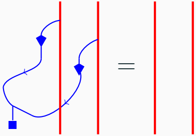
The counit of $\mathcal{O}(G)$ induces a monoidal natural isomorphism

$$(\underline{\mathcal{O}}(G) \otimes -)^G \cong F(-).$$

- The LHS is the composition

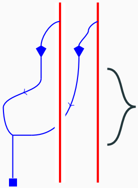
$$\mathcal{YD}_{\mathfrak{g}} \xrightarrow{\text{Free}} \underline{\mathcal{O}}(G)\text{-mod}_{\mathcal{YD}_{\mathfrak{g}}} \xrightarrow{(-)^G} \mathrm{YD}.$$

- Graphically



$\in \mathrm{End}_{\mathrm{YD}}(V^{\otimes 2})$.

- The choice of an associator Φ turns $\mathcal{YD}_{\mathfrak{g}}$ into a braided monoidal category $\mathcal{YD}_{\mathfrak{g}}^{\Phi}$.
- Let

$J_\Phi =$  ⚠ this part now happens in $\mathcal{YD}_{\mathfrak{g}}^{\Phi}$.

Theorem (Etingof–Kazhdan)

$J_\Phi = 1 +$  $+ \ldots$ *is a monoidal structure on* $F : \mathcal{YD}_{\mathfrak{g}}^{\Phi} \to \mathrm{YD}$, *i.e. a (group-like) solution of the twist equation*

$$\Phi J_\Phi^{12,3} J_\Phi^{1,2} = J_\Phi^{1,23} J_\Phi^{2,3} \in \mathrm{End}(V^{\otimes 3}).$$

"Bar Natan's dream"

If we further impose that $\mathfrak{g}$ and V are dualizable/finite-dimensional, one gets a homomorphic expansion for virtual tangles.

Key observation

J_Φ acts trivially on G-invariant by construction!

$$\begin{array}{ccc} V^G \otimes V^G & \longrightarrow & (V \otimes V)^G \\ \downarrow & & \downarrow \\ (\underline{\mathcal{O}}(G) \otimes V)^G \otimes (\underline{\mathcal{O}}(G) \otimes V)^G & \xrightarrow{J_\Phi} & (\underline{\mathcal{O}}(G) \otimes V \otimes V)^G \end{array}$$

- When $\mathfrak{g}$ is f.d. one can run EK's construction with $G^\vee$ instead of G: this produces another solution $J_\Phi^\vee$ of the twist equation in $\mathrm{End}(V^{\otimes 2})$.
- $J_\Phi^\vee = \tau(J_\Phi)^{-1}$ where τ is a rather explicit anti-equivalence of YD which lift $\mathfrak{g} \mapsto \mathfrak{g}^\vee$.
- One gets two quantizations of $G^\vee$:
  - the functorial one $\mathcal{O}_\Phi(G)$ which uses $J_\Phi^\vee$,
  - another one where the product is given by the action of J_Φ.
- In the latter case the product is "of Duflo type" on the nose, i.e. it's undeformed on the G-invariant part.
- But EK construction is "compatible with duality" (Enriquez–Geer), hence those two quantizations are isomorphic: this implies "Duflo's theorem".

- In turn this follows from the fact that these solutions of the twist equation are equivalent: there exists an invertible $u \in \mathrm{End}(V)$ such that

$$u_{V\otimes V} J_\Phi = J_\Phi^\vee (u \otimes u)$$

- In fact *any two* solutions of the twist equation are equivalent in that sense (provided they have the same quasi-classical limit).
- In other words, u induces a monoidal natural isomorphism

$$(F, J_\Phi) \cong (F, J_\Phi^\vee).$$

- Writing $\tilde{\otimes}$ for the tensor product conjugated by $J_\Phi^\vee$ and writing $\bar{\ }$ for the restriction of the action on $V^G \otimes V^G$, one gets the following:

"Kashiwara–Vergne"

There exists an invertible $u \in \mathrm{End}(V)$ such that

$$\overline{J_\Phi^\vee} = (\bar{u} \tilde{\otimes} \bar{u})^{-1} \overline{u_{V\otimes V}}.$$

- In the f.d. case one can normal order to put the actions in front of coactions, so on invariants what's left afterwards are only "wheels with tails".

- In the Lie algebra case, i.e. if  $= 0$, I believe this essentially recovers Alekseev–Torossian's divergence and its role in Duflo's theorem: if the divergence vanishes, then the action on invariant is trivial.

- In general however I don't expect this to be a well-defined operation: in the Lie algebra case the map from acyclic to arbitrary diagrams splits (see [WKO2] for more on this)

- The braided monoidal category $\mathcal{YD}_{\mathfrak{g}}^{\Phi}$ is the value on the disk of a certain framed 2d TFT [2].
- The commutative algebra $\underline{\mathcal{O}}(G)$ (rather its monoidal category of modules) defines a boundary condition, aka a Swiss–Cheese algebra:

- This TFT is compatible with (framed) embeddings: the monoidal structure on $\underline{\mathcal{O}}(G)\text{-mod}_{\mathcal{YD}_{\mathfrak{g}}^{\Phi}}$ is given by stacking vertically, and the free module functor by

- There is also a lift of $S(\mathfrak{g}^*)$ to a cocommutative coalgebra in that category, which defines a left boundary condition

- The "fundamental theorem of Hopf algebras" is the statement that the following inclusion

$$\text{Vect} \simeq \emptyset \hookrightarrow$$ 

is a monoidal equivalence.

[2] See also Lie-Bland–Ševera and Safronov

- The fiber functor, aka the solution of the twist equation, comes from

- Note this is isotopic to

- If $\mathfrak{g}$ is finite dimensional
  - the theory becomes oriented,
  - $\mathcal{YD}_{\mathfrak{g}} \simeq \mathfrak{d}$-mod where $\mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^{\vee}$ is the so-called Drinfeld double,
  - and we have Poisson isos

$$D/G^{\vee} \cong G \qquad\qquad G\backslash D \cong G^{\vee}.$$

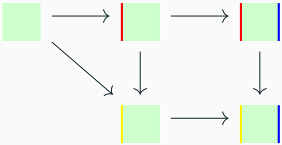
- There is another boundary condition associated with $\underline{\mathcal{O}}(G^{\vee})$

- Then

 $\simeq \mathrm{Vect} \longleftrightarrow G\backslash D/G^{\vee} \cong pt.$

- The compatibility of EK with duality may be be thought of as coming from a morphism of SC algebras

- since then the diagram



commutes.
- The double quotient $G\backslash D/G \cong G^\vee/G$ has the zero Poisson bracket (really a 1-shifted Poisson structure in the sense of [CPTVV].
- The undeformed algebra $\mathcal{O}(G^\vee)^G$ is recovered as endomorphism of the unit in the monoidal category

aka the skein algebra of that surface.
- Finally, Duflo is the quantization of the Poisson morphism $G^\vee \to G^\vee/G$ induced by

Thank you for your attention !