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QUASI-HOPF ALGEBRAS

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ABSTRACT. The author introduces and investigates the notion of quasi-Hopf algebra, obtained from that of Hopf algebra by a weakening of the coassociativity axiom. The weakened axiom still ensures associativity of the tensor product of representations. In the context of quasi-Hopf algebras the notions of classical limit and of quantization are examined, the structure of certain classes of quasi-Hof algebras are described, and the connections with conformal field theory and knot invariants are discussed.

We recall that a Hopf algebra is a pair (A, Δ), where A is an associative algebra and Δ a homomorphism $A \to A \otimes A$ satisfying certain conditions, principal among which is the coassociativity of Δ, i.e., equality of the mappings $(\Delta \otimes \mathrm{id}) \circ \Delta: A \to A \otimes A \otimes A$ and $(\mathrm{id} \otimes \Delta) \circ \Delta: A \to A \otimes A \otimes A$. The existence of Δ permits the definition of the tensor product of two representations M_1 and M_2 of the algebra A, and the coassociativity of Δ implies the existence of a natural isomorphism $(M_1 \otimes M_2) \otimes M_3 \xrightarrow{\sim} M_1 \otimes (M_2 \otimes M_3)$. If, in addition, Δ is cocommutative, i.e., if $\Delta = \Delta'$, where Δ' is the composite of Δ and the mapping $A \otimes A \to A \otimes A$ that switches factors, then there exists a natural isomorphism $M_1 \otimes M_2 \xrightarrow{\sim} M_2 \otimes M_1$. The development of the quantum method for the inverse problem [1] has led (see [2]) to the notion of quasitriangular Hopf algebra. This is a triple (A, Δ, R), where (A, Δ) is a Hopf algebra, $R \in A \otimes A$, $\Delta'(a) = R\Delta(a)R^{-1}$ for $a \in A$, and the "self-consistency" relations (3.2) are satisfied, from which follows the Yang-Baxter equation (3.4). In this case we have as before a natural isomorphism $M_1 \otimes M_2 \xrightarrow{\sim} M_2 \otimes M_1$ (in whose definition R plays a part), while (3.2) ensures the commutativity of certain natural diagrams (see (3.5)). In other words, the representations of A form a *quasitensor category*. In accordance with Reshetikhin [3], it is this property of A (together with the notion of a contragredient representation, defined by means of the antipode $S: A \to A$) that allows us to associate with every knot in $\mathbb{R}^3$ an element of the center of A, generalizing the Jones polynomial [4].

It is natural to weaken the coassociativity condition on Δ by replacing it with the equailty $(\mathrm{id} \otimes \Delta)(\Delta(a)) = \Phi \cdot (\Delta \otimes \mathrm{id})(\Delta(a)) \cdot \Phi^{-1}$, $a \in A$, where $\Phi \in A \otimes A \otimes A$ must satisfy a natural self-consistency condition (see (1.2)). This leads to the notion of quasi-Hopf algebra (§1) and the notions of quasitriangular, triangular, and coboundary quasi-Hopf algebra (§3). Since the representations of a quasitriangular quasi-Hopf algebra form a quasitensor category, the method

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of Reshetikhin [3] for constructing knot invariants generalizes to the quasi-Hopf cases.

In accordance with the "philosophy" of quantum groups [2], along with Hopf algebras (the "quantum" objects) it is useful to consider their "classical" analogues: Lie bialgebras. Specification of a Lie bialgebra is equivalent to specification of a triple $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$, where $\mathfrak{p}$ is a Lie algebra with an invariant scalar product, and $\mathfrak{p}_1, \mathfrak{p}_2 \subset \mathfrak{p}$ are transversal Lagrangian subalgebras. It turns out (see §§2, 3) that in the quasi-Hopf case the role of such triples is played by pairs $(\mathfrak{p}, \mathfrak{p}_1)$; and in the quasitriangular quasi-Hopf case by pairs $(\mathfrak{g}, t)$, where $\mathfrak{g}$ is a Lie algebra and $t \in \mathrm{Sym}^2\mathfrak{g}$ an invariant tensor. It is likely that also in the quantum situation the quasi-Hopf algebras will be simpler than the Hopf, thanks to the existence in the quasi-Hopf case of "gauge" transformations, called *twists* (see §1). In any case, the quasitriangular quasi-Hopf algebras within the scope of perturbation theory in Planck's constant h admit a simple description: by Theorem 3.15, they are in one-to-one correspondence, up to twist, with their classical analogues $(\mathfrak{g}, t)$, where the structural constants of $\mathfrak{g}$ and the components of t depend on h. For lack of space, we must omit here the proof of Theorem 3.15; but we indicate how to construct, for a given $(\mathfrak{g}, t)$, a quasitriangular quasi-Hopf algebra $A_{\mathfrak{g},t} = (A, \Delta, \Phi, R)$. Indeed, A is the universal enveloping algebra, Δ the usual comultiplication, $R = e^{ht/2}$, and Φ is defined by means of the Knizhnik-Zamolodchikov system of equations

$$\frac{\partial W}{\partial z_i} = \frac{h}{2\pi i} \sum_{j \neq i} \frac{t^{ij}}{z_i - z_j} \cdot W, \qquad 1 \leq i \leq n, \tag{0.1}$$

which is satisfied by the correlation functions in the Wess-Zumino-Witten model of conformal field theory (t^{ij} is the image of t under the (i, j)th imbedding $\mathfrak{g} \otimes \mathfrak{g} \to (U\mathfrak{g})^{\otimes n}$). Specifically, $\Phi = W_2^{-1} W_1$, where W_1 and W_2 are solutions of the system (0.1) in the domain $z_1 > z_2 > z_3$ having standard asymptotic behavior for $z_1 - z_2 \ll z_2 - z_3$ and $z_1 - z_2 \gg z_2 - z_3$, respectively (see §3 for details). This construction leads (see the end of §3) to a natural proof of the important theorem of Kohno [5], asserting that if $\mathfrak{g}$ is finite-dimensional and semisimple, and ρ is a finite-dimensional representation of $\mathfrak{g}$, then the representation of the braid group B_n determined by the monodromy of the equation

$$\frac{\partial W}{\partial z_i} = \frac{h}{2\pi i} \sum_{j \neq i} \frac{\rho^{\otimes n}(t^{ij})}{z_i - z_j} \cdot W, \qquad 1 \leq i \leq n, \tag{0.2}$$

is equivalent to the representation of B_n constructed with respect to a certain R-matrix.

It was in fact this theorem of Kohno, together with earlier results [6]–[8], that suggested to the present author the construction of $A_{\mathfrak{g},t}$ by means of (0.1). The argument ran as follows. If we are given a representation of the quasitriangular quasi-Hopf algebra A in a space V, then there is an action of B_n in $V^{\otimes n}$ (this follows from the fact that the representations of A form a quasitensor category C; if C were tensor, i.e., if the commutativity isomorphism $V_1 \otimes V_2 \xrightarrow{\sim} V_2 \otimes V_1$, $V_i \in C$, were involutory, then $V^{\otimes n}$ would have an action of the permutation group S_n). Kohno's theorem suggested that the quasitriangular quasi-Hopf algebra $A_{\mathfrak{g},t}$ (for which the definition was at that point unknown) should have the following properties: 1) $A_{\mathfrak{g},t}$ should be the universal envelope with the usual comuliplication, but with nontrivial R and Φ; 2) if ρ is a representation of $A_{\mathfrak{g},t}$ in a space V, then the corresponding representation of B_n in $V^{\otimes n}$ should be defined by the monodromy of the system (0.2). Property 2) afforded a guess at R and Φ.

Besides the results listed above, the present paper contains (see §3) a proof of the analogues of Theorem 3.15 for triangular and coboundary quasi-Hopf algebras, as well as an outline of the construction of a knot invariant that contains all the invariants of R-matrix type (see [3] and the literature cited there) corresponding to quasiclassical R-matrices.

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§1. Definition and elementary properties of quasi-Hopf algebras

We recall that by a bialgebra over a commutative ring k is meant an associative k-algebra A with unity, provided with homomorphisms $\Delta: A \to A \otimes A$ and $\varepsilon: A \to k$ such that $(\varepsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id}$ and satisfying the coassociativity condition $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ (both sides of this equality are homomorphisms $A \to A \otimes A \otimes A$). It is understood that Δ and ε are homomorphisms of an algebra with unity; i.e., $\Delta(1) = 1$ and $\varepsilon(1) = 1$.

By A-mod we denote the category of (left) A-modules. If A is a bialgebra, there exists, as is well known, a functor $\otimes$: (A-mod) $\times$ (A-mod) $\to$ A-mod as follows: if M and N are modules over A, then $M \otimes N$ is $M \otimes_k N$ with the A-module structure defined by the homomorphism $\Delta: A \to A \otimes A$. The pair ($A$-mod, $\otimes$) is a monoidal category [9] (in general, nonsymmetric, since the A-modules $M \otimes N$ and $N \otimes M$ need not be isomorphic). In this monoidal category the associativity morphism (= associativity constraint) is trivial, and the identity object is k with the A-module structure defined by the homomorphism ε. It turns out that this notion of bialgebra can be generalized so that the category A-mod is still monoidal, but with nontrivial associativity morphism.

Definition. By a quasibialgebra is meant a set $(A, \Delta, \varepsilon, \Phi)$, where A is an associative k-algebra with unity, Δ a homomorphism $A \to A \otimes A$, ε a homomorphism $A \to k$, and Φ an invertible element of $A \otimes A \otimes A$, such that the following equalitites hold:

$$(\mathrm{id} \otimes \Delta)(\Delta(a)) = \Phi \cdot (\Delta \otimes \mathrm{id})(\Delta(a)) \cdot \Phi^{-1}, \qquad a \in A, \tag{1.1}$$

$$(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi) \cdot (\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\Phi) = (1 \otimes \Phi) \cdot (\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Phi) \cdot (\Phi \otimes 1), \tag{1.2}$$

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta, \tag{1.3}$$

$$(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(\Phi) = 1. \tag{1.4}$$

If A is a quasibialgebra, the tensor product of A-modules and the identity A-module are defined in the same way as for bialgebras. From (1.1) it follows that for any A-modules M_1, M_2, M_3 the mapping $\varphi: (M_1 \otimes M_2) \otimes M_3 \to M_1 \otimes (M_2 \otimes M_3)$, defined as the image of Φ in $\mathrm{End}_k(M_1 \otimes M_2 \otimes M_3)$, is an A-module isomorphism; and from (1.3) it follows that the natural mappings $M \to k \otimes M$ and $M \to M \otimes k$ are also A-module isomorphisms. From (1.2) and (1.4) it follows that the coherence conditions (see [9]–[11]) are satisfied;

i.e., that the diagrams

$$\begin{array}{ccc} ((M_1 \otimes M_2) \otimes M_3) \otimes M_4 \to (M_1 \otimes M_2) \otimes (M_3 \otimes M_4) & \to & M_1 \otimes (M_2 \otimes (M_3 \otimes M_4)) \\ \downarrow & & \downarrow \\ (M_1 \otimes (M_2 \otimes M_3)) \otimes M_4 & \longrightarrow & M_1 \otimes ((M_2 \otimes M_3) \otimes M_4) \end{array} \tag{1.5}$$

$$\begin{tikzcd} & M_1 \otimes M_2 \arrow[dl] \arrow[dr] & \\ (M_1 \otimes k) \otimes M_2 \arrow[rr] & & M_1 \otimes (k \otimes M_2) \end{tikzcd} \tag{1.6}$$

are commutative. Thus, the category A-mod is monoidal.

To make formulas like (1.1)–(1.4) more expressive, we introduce some new notation. Let us think of A as an algebra of functions on a "noncommutative space" X. An element $a \in A$ will be written as $a(x)$, an element $b \in A \otimes A$ as $b(x, y)$, etc., where x, y, etc., are points of X. If $a \in A$, then instead of $\Delta(a)$ we write $a(x * y)$, where $*$ is an "operation" on X (i.e., a "mapping" $X \times X \to X$). The homomorphism $\varepsilon: A \to k$ determines a point of X, which we denote by 0; thus, instead of $\varepsilon(a)$ we write $a(0)$. Then formulas (1.1)–(1.4) can be rewritten as

$$a(x * (y * z)) = \Phi(x, y, z) a((x * y) * z) \Phi(x, y, z)^{-1}, \qquad a \in A, \tag{1.7}$$

$$\Phi(x, y, z * u) \cdot \Phi(x * y, z, u) = \Phi(y, z, u) \cdot \Phi(x, y * z, u) \cdot \Phi(x, y, z), \tag{1.8}$$

$$a(0 * x) = a(x) = a(x * 0), \qquad a \in A, \tag{1.9}$$

$$\Phi(x, 0, z) = 1. \tag{1.10}$$

REMARK. From (1.8)–(1.10) it follows that $\Phi(0, y, z) = 1 = \Phi(x, y, 0)$ (for example, putting $x = y = 0$ in (1.8), we obtain $\Phi(0, y, z) = 1$). This is an analogue of Kelly's theorem [11] that commutativity of (1.5) and (1.6) implies commutativity of the diagrams

$$\begin{tikzcd} (k \otimes M_1) \otimes M_2 \arrow[rr] \arrow[dr] & & k \otimes (M_1 \otimes M_2) \arrow[dl] \\ & M_1 \otimes M_2 & \end{tikzcd} \qquad \begin{tikzcd} (M_1 \otimes M_2) \otimes k \arrow[rr] \arrow[dr] & & M_1 \otimes (M_2 \otimes k) \arrow[dl] \\ & M_1 \otimes M_2 & \end{tikzcd}$$

Suppose given a quasibialgebra $(A, \Delta, \varepsilon, \Phi)$ and an invertible element $F \in A \otimes A$ such that $F(x, 0) = 1 = F(0, y)$. Put

$$\tilde{\Delta}(a) = F \Delta(a) F^{-1}, \tag{1.11}$$

$$\tilde{\Phi}(x, y, z) = F(y, z) F(x, y * z) \Phi(x, y, z) \times F(x * y, z)^{-1} F(x, y)^{-1}, \tag{1.12}$$

where $*$ corresponds to Δ (and not to $\tilde{\Delta}$). Then $(A, \tilde{\Delta}, \varepsilon, \tilde{\Phi})$ is also a quasibialgebra. We say that $(A, \tilde{\Delta}, \varepsilon, \tilde{\Phi})$ is obtained from $(A, \Delta, \varepsilon, \Phi)$ by *twisting via the element* F. Twisting via $F_1 F_2$ is equivalent to twisting first via F_2, then via F_1.

If $(A, \tilde{\Delta}, \varepsilon, \tilde{\Phi})$ is obtained from $(A, \Delta, \varepsilon, \Phi)$ by twisting via F, then the corresponding monoidal categories C and $\tilde{C}$ are equivalent (in a monoidal sense). To see this, we need to construct a category equivalence $f: C \to \tilde{C}$, a natural transformation $\mathscr{F}: f(M_1 \otimes M_2) \xrightarrow{\sim} f(M_1) \otimes f(M_2)$, and an isomorphism $\omega: f(k) \xrightarrow{\sim} k$ such that the diagrams

$$\begin{array}{ccc} f((M_1 \otimes M_2) \otimes M_3) & \xrightarrow{\sim} & (f(M_1) \otimes f(M_2)) \otimes f(M_3) \\ \downarrow \wr & & \downarrow \wr \\ f(M_1 \otimes (M_2 \otimes M_3)) & \xrightarrow{\sim} & f(M_1) \otimes (f(M_2) \otimes f(M_3)) \end{array}$$

$$\begin{array}{ccc} f(k\otimes M) & \xrightarrow{\sim} & k\otimes f(M) \\ \uparrow\wr & \nearrow\sim & \\ f(M) & & \end{array} \qquad \begin{array}{ccc} & & f(M) \\ & \swarrow\sim & \downarrow\wr \\ f(M\otimes k) & \xrightarrow{\sim} & f(M)\otimes k \end{array}$$

are commutative. It suffices to put $\omega, f = \mathrm{id}$ (if we forget the monoidal structure, then $C = \tilde{C} = (A\text{-mod})$), and for $\mathcal{F}: M_1 \otimes M_2 \to M_1 \otimes M_2$ take the image of F in $\mathrm{End}_k(M_1 \otimes M_2)$.

REMARKS. 1) It might be more natural to define a quasibialgebra as a set $(A, \Delta, \varepsilon, \Phi, f, g)$ where $A, \Delta, \varepsilon, \Phi$ are as before, f and g are invertible elements of A, and in addition to (1.7), (1.8) the following analogues of (1.9), (1.10) are satisfied:

$$a(0 * x) = f(x)a(x)f(x)^{-1}, \tag{1.13}$$

$$a(x * 0) = g(x)a(x)g(x)^{-1}, \tag{1.14}$$

$$\Phi(x, 0, z) = g(x)^{-1}f(z). \tag{1.15}$$

Then for any A-module M the images of f and g in $\mathrm{End}_k M$ define A-module isomorphisms $M \to k\otimes M$ and $M \to M \otimes k$, respectively, while (1.15) ensures commutativity of the diagram (1.6). Hence A-mod is a monoidal category. With this definition of quasibialgebra, a twist is defined by the formulas

$$\tilde{f}(x) = F(0, x)f(x), \qquad \tilde{g}(x) = F(x, 0)g(x) \tag{1.16}$$

together with (1.11), (1.12); it is not required that $F(x, 0) = 1 = F(0, x)$. But the two definitions of quasibialgebra are in fact essentially equivalent. Indeed, if $A, \Delta, \varepsilon, \Phi, f, g$ satisfy (1.7), (1.8) and (1.13)–(1.15), we can always choose an F so that $\tilde{f}$ and $\tilde{g}$, defined by formulas (1.16), are equal to 1 (to see this, we need to verify that $f(0) = g(0)$ and for that it suffices to put $x = z = 0$ in (1.15) and $x = y = z = u = 0$ in (1.8)).

2) Suppose A, Δ, Φ satisfy (1.7), (1.8), and there exist homomorphisms $\varepsilon: A \to k$ and $\varepsilon': A \to k$ such that $(\varepsilon \otimes \mathrm{id}) \circ \Delta$ and $(\mathrm{id} \otimes \varepsilon') \circ \Delta$ are inner automorphisms of A. Then ε and ε' are unique, $\varepsilon' = \varepsilon$, and there exist invertible elements $f, g \in A$ satisfying (1.13)–(1.15), with f and g unique up to replacement by cf, cg, where $c \in k^*$. Indeed, first, the equality $\varepsilon' = \varepsilon$ and the uniqueness of ε are proved by consideration of the homomorphism $(\varepsilon \otimes \varepsilon') \circ \Delta: A \to k$. From (1.15) it follows that if f and g exist, then $f(x) = c\Phi(0, 0, x)$ and $g(x) = c\Phi(x, 0, 0)^{-1}$, where $c \in k^*$ It remains only to show that (1.13)–(1.15) are satisfied if we put $f(x) = \Phi(0, 0, x)$ and $g(x) = \Phi(x, 0, 0)^{-1}$. To prove (1.13) and (1.14), it suffices to put, respectively, $x = y = 0$ and $y = z = 0$ in (1.7); and to prove (1.15), to put $y = z = 0$ in (1.8).

We shall understand by a Hopf algebra a bialgebra A with a bijective antipode $S: A \to A$. This differs somewhat from the conventional definition in [12] (where bijectivity of S is not required), and is equivalent to the definition in [2] (by Heyneman's theorem ([13], p. 56), bijectivity of S is equivalent to existence of a skew antipode, which then equals S^{-1}). We denote by C_A the category of A-modules that are free k-modules of finite type. If A is a Hopf algebra, then C_A is a *rigid* monoidal category. Rigidity means, by definition, that for each object $M \in C_A$ there exist left- and right-dual objects. Here by a left dual for M we mean an object $N \in C_A$ together with morphisms $N \otimes M \to k$ and $k \to M \otimes N$ such that the composites $M \to (M \otimes N) \otimes M \xrightarrow{\sim} M \otimes (N \otimes M) \to M$ and $N \to N \otimes (M \otimes N) \xrightarrow{\sim} (N \otimes M) \otimes N \to N$ are the identity morphisms; and then M is called a right dual for N. We note that in any monoidal category left- and right-dual objects are unique if they exist (proof below), and that for symmetric monoidal categories this definition of rigidity is equivalent to that in §1 in [14]. The dual objects in C_A are constructed as follows: 1) the left-dual object ${}^*M$ is $\operatorname{Hom}_k(M, k)$ with the action of A given by $a \mapsto (\rho(S(a)))^*$, where $a \in A$ and $\rho: A \to \operatorname{End}_k M$ defines the A-module structure on M, while the morphisms ${}^*M \otimes M \to k$ and $k \to M \otimes {}^*M$ are defined in the natural fashion; 2) the right-dual object M^* is constructed similarly, but with S replaced by S^{-1}.

An attempt to define a class of quasibialgebras A for which the monoidal category C_A is rigid leads to the following definition.

Definition. By a quasi-Hopf algebra is meant a quasibialgebra $(A, \Delta, \varepsilon, \Phi)$ for which there exist $\alpha, \beta \in A$ and an antiautomorphism S of A such that

$$\sum_i S(b_i)\alpha c_i = \varepsilon(a)\alpha, \qquad \sum_i b_i \beta S(c_i) = \varepsilon(a)\beta \tag{1.17}$$

for $a \in A$ and $\sum_i b_i \otimes c_i = \Delta(a)$, and

$$\sum_i X_i \beta S(Y_i)\alpha Z_i = 1, \quad \text{where } \sum_i X_i \otimes Y_i \otimes Z_i = \Phi, \tag{1.18}$$

$$\sum_j S(P_j)\alpha Q_j \beta S(R_j) = 1, \quad \text{where } \sum_j P_j \otimes Q_j \otimes R_j = \Phi^{-1}. \tag{1.19}$$

Remarks. 1) If S, α, β satisfy conditions (1.17)–(1.19), then for any invertible $u \in A$ the same conditions are satisfied by $\overline{S}, \overline{\alpha}, \overline{\beta}$, where

$$\overline{S}(a) = uS(a)u^{-1}, \qquad \overline{\alpha} = u\alpha, \qquad \overline{\beta} = \beta u^{-1}. \tag{1.20}$$

2) Let $\Phi = 1$, so that (A, Δ, ε) is a bialgebra. Then (1.18) and (1.19) imply that $\alpha\beta = \beta\alpha = 1$. The transformation (1.20) therefore allows us to suppose, without loss of generality, that $\alpha = \beta = 1$; and then (1.17) becomes the ordinary definition of antipode. Thus, for bialgebras quasi-Hopf is equivalent to Hopf.

3) If A is a quasi-Hopf algebra, then the monoidal category C_A defined above is rigid. The duals ${}^*M$ and M^* are defined as in the Hopf case, but the morphisms $f_1: {}^*M \otimes M \to k$, $f_2: k \to M \otimes {}^*M$, $f_3: M \otimes M^* \to k$, and $f_4: k \to M^* \otimes M$ are now given by the formulas $f_1 = \varphi_1 \circ (\mathrm{id} \otimes \rho(\alpha))$, $f_2 = (\rho(\beta) \otimes \mathrm{id}) \circ \varphi_2$, $f_3 = \varphi_3 \circ (\rho(S^{-1}(\alpha)) \otimes \mathrm{id})$, and $f_4 = (\mathrm{id} \otimes \rho(S^{-1}(\beta))) \circ \varphi_4$, where $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ are the natural mappings $\operatorname{Hom}_k(M, k) \otimes_k M \to k$, $k \to \operatorname{Hom}_k(M, k) \otimes_k M$, etc., and $\rho: A \to \operatorname{End}_k M$ defines the A-module structure on M.

4) To every quasibialgebra $(A, \Delta, \varepsilon, \Phi)$ are connected three others: a) without changing Δ, we can replace multiplication in A by the opposite operation, and Φ by Φ^{-1}; b) without changing multiplication in A, we can replace Δ by $\Delta' = \sigma \circ \Delta$, and $\Phi(x, y, z)$ by $\Phi(z, y, x)^{-1}$, where $\sigma: A \otimes A \to A \otimes A$ switches the factors; c) we can replace multiplication by the opposite operation, Δ by Δ', and $\Phi(x, y, z)$ by $\Phi(z, y, x)$. If $(A, \Delta, \varepsilon, \Phi)$ is quasi-Hopf, then the other three quasibialgebras are also quasi-Hopf: in case a) we must replace S, α, β by $\overline{S} = S^{-1}$, $\overline{\alpha} = S^{-1}(\beta)$, $\overline{\beta} = S^{-1}(\alpha)$; in case b) we must put $\overline{S} = S^{-1}$, $\overline{\alpha} = S^{-1}(\alpha)$, $\overline{\beta} = S^{-1}(\beta)$; and in case c), $\overline{S} = S$, $\overline{\alpha} = \beta$, $\overline{\beta} = \alpha$.

5) The property of being quasi-Hopf is preserved under a twist. Indeed, if $(A, \Delta, \varepsilon, \Phi)$ is quasi-Hopf, S, α, β satisfy (1.17)–(1.19), and $\tilde{\Delta}$ and $\tilde{\Phi}$ are defined by (1.11), (1.12), then the roles of S, α, β for $(A, \tilde{\Delta}, \varepsilon, \tilde{\Phi})$ are played by $\tilde{S} = S$, $\tilde{\alpha} = \sum_i S(d_i)\alpha e_i$, $\tilde{\beta} = \sum_j f_j \beta S(g_j)$, where $\sum_i d_i \otimes e_i = F^{-1}$ and $\sum_j f_j \otimes g_j = F$.

6) In the definition of quasi-Hopf algebra, before the words "there exist" should probably be inserted "locally with respect to $\operatorname{Spec} k$". We are not doing this, because we are primarily concerned with the case that k is a field or the ring of formal series over a field.

7) From (1.17) and (1.18) it follows that $\varepsilon \circ S = \varepsilon$. Indeed, applying $\varepsilon \otimes \varepsilon$ to (1.17), we obtain $\varepsilon(S((a))\alpha = \varepsilon(a)\alpha$, and (1.18) implies that $\varepsilon(\alpha)$ is invertible.

8) It will be shown (Proposition 1.3) that either one of conditions (1.18) and (1.19) is superfluous. But the redundant system of axioms (1.17)–(1.19) is more symmetric (see Remark 4).

Proposition 1.1. *If two triples* (S, α, β) *and* $(\overline{S}, \overline{\alpha}, \overline{\beta})$ *satisfy* (1.17)–(1.19), *then they are connected by a transformation* (1.20), *where* u *is uniquely determined.*

Proof. If (S, α, β) and $(\overline{S}, \overline{\alpha}, \overline{\beta})$ are connected by (1.20), then

$$u = u \sum_i S(P_i)\alpha Q_i \beta S(R_i) = \sum_i \overline{S}(P_i)\overline{\alpha} Q_i \beta S(R_i),$$

where $\sum_i P_i \otimes Q_i \otimes R_i = \Phi^{-1}$. Conversely, putting $u = \sum_i \overline{S}(P_i)\overline{\alpha} Q_i \beta S(R_i)$, let us prove (1.20). If $a \in A$, then $uS(a) = \overline{S}(a)u$: it suffices to apply to both sides of the equality $(\Delta \otimes \mathrm{id})(\Delta(a)) \cdot \Phi^{-1} = \Phi^{-1} \cdot (\mathrm{id} \otimes \Delta)(\Delta(a))$ the k-linear mapping $A \otimes A \otimes A \to A$ taking $b \otimes c \otimes d$ into $\overline{S}(b)\overline{\alpha} c \beta S(d)$. Next, $u\alpha = \overline{\alpha}$; it suffices to apply to both sides of the equality

$$\begin{aligned} &(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi) \cdot (\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\Phi) \cdot (\Phi^{-1} \otimes 1) \\ &\quad = (1 \otimes \Phi) \cdot (\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Phi), \end{aligned} \tag{1.21}$$

which is equivalent to (1.2), the k-linear mapping $V: A \otimes A \otimes A \otimes A \to A$ taking $b \otimes c \otimes d \otimes e$ into $\overline{S}(b)\overline{\alpha} c \beta S(d)\alpha e$, and to use the fact that

$$\begin{aligned} V((\Delta(a) \otimes 1 \otimes 1) \cdot T) &= \varepsilon(a)V(T) = V((1 \otimes 1 \otimes \Delta(a)) \cdot T) \\ &= V(T \cdot (1 \otimes \Delta(a) \otimes 1)) \end{aligned}$$

for $a \in A$ and $T \in A \otimes A \otimes A \otimes A$. That $\overline{\beta} u = \beta$ follows from the equality $u\alpha = \overline{\alpha}$ if we replace multiplication in A by the opposite operation and simultaneously change Δ to Δ' and $\Phi(x, y, z)$ to $\Phi(z, y, x)$ (see Remark 4 to the definition of quasi-Hopf algebra). Finally, we construct an inverse element to u. It is $v = \sum_i S(P_i)\alpha Q_i \overline{\beta}\, \overline{S}(R_i)$. Indeed,

$$\begin{aligned} uv &= \sum_i uS(P_i)\alpha Q_i \overline{\beta}\, \overline{S}(R_i) = \sum_i \overline{S}(P_i) u\alpha Q_i \overline{\beta}\, \overline{S}(R_i) \\ &= \sum_i \overline{S}(P_i)\overline{\alpha} Q_i \overline{\beta}\, \overline{S}(R_i) = 1; \end{aligned}$$

and similarly $vu = 1$. ●

Remark. This proof of Proposition 1.1 results from an analysis of the following proof of the uniqueness of the left-dual object in a monoidal category.

Let ${}^*M$ and ${}^+M$ be two left-dual objects for M. Consider the composite ${}^+M \to {}^+M \otimes (M \otimes {}^*M) \xrightarrow{\sim} ({}^+M \otimes M) \otimes {}^*M \to {}^*M$ and the analogous morphism ${}^*M \to {}^+M$ (computing these morphisms for a module M over the algebra A of Proposition 1.1, with ${}^*M$ and ${}^+M$ corresponding to S and $\overline{S}$, one easily arrives at the formulas for u and v in the proof of the proposition). It remains to prove that the two morphisms are mutual inverses and that the diagrams

$$\begin{tikzcd} {}^+M \otimes M \arrow[r] \arrow[dr] & {}^*M \otimes M \arrow[d] \\ & k \end{tikzcd} \qquad \begin{tikzcd} & k \arrow[dl] \arrow[dr] & \\ M \otimes {}^+M \arrow[rr] & & M \otimes {}^*M \end{tikzcd} \tag{1.22}$$

are commutative. Commutativity of, say, the left-hand diagram in (1.22) follows from a consideration of the diagram

$$\begin{tikzcd} ({}^+M \otimes (M \otimes {}^*M)) \otimes M \arrow[rr, "\sim"] \arrow[d, "\wr"] & & (({}^+M \otimes M) \otimes {}^*M) \otimes M \arrow[d] \arrow[ddr, bend left, "\wr"] & \\ {}^+M \otimes ((M \otimes {}^*M) \otimes M) \arrow[d, "\wr"] & {}^+M \otimes M \arrow[l] \arrow[dr] & {}^*M \otimes M \arrow[d] & \\ {}^+M \otimes (M \otimes ({}^*M \otimes M)) \arrow[rrr, "\sim"] & & k & ({}^+M \otimes M) \otimes ({}^*M \otimes M) \arrow[l] \end{tikzcd} \tag{1.23}$$

Observe that the derivation of the formula $u\alpha = \overline{\alpha}$ (see the proof of Proposition 1.1) is obtained by analysis of (1.23) (in particular, (1.21) corresponds to the pentagon of isomorphisms in (1.23)). Finally, the fact that the composite ${}^+M \to {}^*M \to {}^+M$ is the identity follows from consideration of the diagram

$$\begin{tikzcd} {}^+M \arrow[r] \arrow[d] & {}^+M \otimes (M \otimes M^+) \arrow[r, "\sim"] \arrow[d] & ({}^+M \otimes M) \otimes M^+ \arrow[d] \arrow[dr] & \\ {}^*M \arrow[r] & {}^*M \otimes (M \otimes M^+) \arrow[r, "\sim"] & ({}^*M \otimes M) \otimes M^+ \arrow[r] & M^+ \end{tikzcd}$$

Of course, it would be nice to prove a metatheorem that the validity of a theorem of a certain type concerning monoidal categories implies the validity of an analogous theorem concerning quasibialgebras.

It is known [12] that if (A, Δ) is a Hopf algebra, then the antipode $S\colon A \to A$ is an antiautomorphism with respect not only to multiplication but also to comultiplication. In other words, $(S \otimes S)(\Delta'(S^{-1}(a)) = \Delta(a)$ for $a \in A$, where $\Delta'\colon A \to A \otimes A$ is the opposite comultiplication. Now let $(A, \Delta, \varepsilon, \Phi)$ be a quasi-Hopf algebra, and suppose S, α, β satisfy (1.17)–(1.19). Put $\tilde{\Delta}(a) = (S \otimes S)(\Delta'(S^{-1}(a)))$ and $\tilde{\Phi} = (S \otimes S \otimes S)(\Phi^{321})$, where Φ^{321} is the image of Φ under the mapping $a \otimes b \otimes c \mapsto c \otimes b \otimes a$. Then $(A, \tilde{\Delta}, \varepsilon, \tilde{\Phi})$ is a quasi-Hopf algebra, and is isomorphic to $(A^0, \Delta', \varepsilon, \Phi^{321})$, where A^0 is the algebra opposite to A (the isomorphism $A^0 \xrightarrow{\sim} A$ is S).

PROPOSITION 1.2. *$(A, \tilde{\Delta}, \varepsilon, \tilde{\Phi})$ is obtained from $(A, \Delta, \varepsilon, \Phi)$ by a twist.*

The proof below is analogous to the proof of the following assertion concerning rigid monoidal categories: there exists a functorial isomorphism $({}^*N \otimes {}^*M)^* \xrightarrow{\sim} M \otimes N$ compatible with the associativity morphism.

PROOF. Define $\gamma, \delta \in A \otimes A$ by the formulas

$$\gamma = \sum_i S(U_i)\alpha V_i \otimes S(T_i)\alpha W_i, \tag{1.24}$$

$$\delta = \sum_j K_j \beta S(N_j) \otimes L_j \beta S(M_j), \tag{1.25}$$

where

$$\sum_i T_i \otimes U_i \otimes V_i \otimes W_i = (1 \otimes \Phi^{-1}) \cdot (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi),$$

$$\sum_j K_j \otimes L_j \otimes M_j \otimes N_j = (\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\Phi) \cdot (\Phi^{-1} \otimes 1).$$

The meaning, for example, of the formula for γ is that if M and N are modules over A that are free k-modules of finite type, then the morphism $({}^*N \otimes {}^*M) \otimes (M \otimes N) \to k$, which is easily defined in any monoidal category, takes $\lambda \otimes \mu \otimes x \otimes y$ into $(\mu \otimes \lambda)(\gamma(x \otimes y))$. ●

LEMMA 1.

1)

$$\gamma = \sum_i S(U_i')\alpha V_i' \otimes S(T_i')\alpha W_i'$$

and

$$\delta = \sum_j K_j'\beta S(N_j') \otimes L_j'\beta S(M_j'),$$

where

$$\sum_i T_i' \otimes U_i' \otimes V_i' \otimes W_i' = (\Phi \otimes 1) \cdot (\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\Phi^{-1})$$

and

$$\sum_j K_j' \otimes L_j' \otimes M_j' \otimes N_j' = (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi^{-1}) \cdot (1 \otimes \Phi).$$

2) *If* $a \in A$ *and* $\Delta(a) = \sum_i b_i \otimes c_i$, *then*

$$\sum_i (S \otimes S)(\Delta'(b_i)) \cdot \gamma \cdot \Delta(c_i) = \varepsilon(a)\gamma, \tag{1.26}$$

$$\sum_i \Delta(b_i) \cdot \delta \cdot (S \otimes S)(\Delta'(c_i)) = \varepsilon(a)\delta. \tag{1.27}$$

3) *If* $\sum_i X_i \otimes Y_i \otimes Z_i = \Phi$ *and* $\sum_j P_j \otimes Q_j \otimes R_j = \Phi^{-1}$, *then*

$$\sum_i \Delta(X_i) \cdot \delta \cdot (S \otimes S)(\Delta'(Y_i)) \cdot \gamma \cdot \Delta(Z_i) = 1, \tag{1.28}$$

$$\sum_j (S \otimes S)(\Delta'(P_j)) \cdot \gamma \cdot \Delta(Q_j) \cdot \delta \cdot (S \otimes S)(\Delta'(R_j)) = 1. \tag{1.29}$$

PROOF. 1) Define on $A \otimes A$ the structure of a right module over $A \otimes A \otimes A \otimes A$ by putting $(a \otimes b) \circ (c \otimes d \otimes e \otimes f) = S(d)ae \otimes S(c)bf$. Then

$$\gamma = (\alpha \otimes \alpha) \circ [(1 \otimes \Phi^{-1}) \cdot (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi)].$$

By (1.2),

$$\gamma = (\alpha \otimes \alpha) \circ [(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Phi) \cdot (\Phi \otimes 1) \cdot (\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\Phi^{-1})].$$

Since $(\alpha \otimes \alpha) \circ (a \otimes \Delta(b) \otimes c) = \varepsilon(b)(\alpha \otimes \alpha) \circ (a \otimes 1 \otimes c)$, we have $\gamma = (\alpha \otimes \alpha) \circ [(\Phi \otimes 1) \cdot (\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\Phi^{-1})]$, as required. Similarly for the formula for δ.

2) The left-hand side of (1.26) is equal to $\gamma \circ [(\Delta \otimes \Delta)(\Delta(a))]$, which is equal to

$$(\alpha \otimes \alpha) \circ [(1 \otimes \Phi^{-1}) \cdot (\mathrm{id} \otimes \mathrm{id} \otimes (\Delta)(\Phi) \cdot (\Delta \otimes \Delta)(\Delta(a))].$$

To prove (1.26), it remains to observe that

$$(1 \otimes \Phi^{-1}) \cdot (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi) \cdot (\Delta \otimes \Delta)(\Delta \cdot (a))$$
$$= [(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)(\Delta(a))] \cdot (1 \otimes \Phi^{-1}) \cdot (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi)$$

in view of (1.1), and that

$$(\alpha \otimes \alpha) \circ [(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)(\Delta(a))] = \varepsilon(a)(\alpha \otimes \alpha).$$

Similarly for (1.27).

3) The left-hand side of (1.28) is equal to $\varphi(r)$, where

$$r = \left(1 \otimes 1 \otimes \sum_k T_k \otimes U_k \otimes V_k \otimes W_k\right) \cdot (\Delta \otimes \Delta \otimes \Delta)(\Phi)$$
$$\times \sum_j K_j \otimes L_j \otimes M_j \otimes N_j \otimes 1 \otimes 1 \in A^{\otimes 6}$$

and φ is the k-linear mapping $A^{\otimes 6} \to A \otimes A$ such that

$$\varphi(a \otimes b \otimes c \otimes d \otimes e \otimes f)$$
$$= (a \otimes b) \cdot (\beta \otimes \beta) \cdot (S(d) \otimes S(c)) \cdot (\alpha \otimes \alpha) \cdot (e \otimes f).$$

Using the notational system (1.7)–(1.10), we obtain

$$r(x, y, z, u, v, w) = \Phi(u, v, w)^{-1}\Phi(z, u, v * w)$$
$$\times \Phi(x * y, z * u, v * w)\Phi(x * y, z, u)\Phi(x, y, z)^{-1}.$$

From (1.8) and (1.7) it follows that

$$\begin{aligned} r(x, y, z, u, v, w) &= \Phi(u, v, w)^{-1}\Phi(x * y, z, u * (v * w)) \\ &\quad \times \Phi((x * y) * z, u, v * w)\Phi(x, y, z)^{-1} \\ &= \Phi(x * y, z, (u * v) * w)\Phi(u, v, w)^{-1} \\ &\quad \times \Phi(x, y, z)^{-1}\Phi(x * (y * z), u, v * w). \end{aligned}$$

Since

$$\varphi(h \cdot \Delta^{23}(a)) = \varepsilon(a)h = \varphi(\Delta^{45}(a) \cdot h) \tag{1.30}$$

for $h \in A^{\otimes 6}$ and $a \in A$, where $\Delta^{23}(a) = 1 \otimes \Delta(a) \otimes 1 \otimes 1 \otimes 1$ and $\Delta^{45}(a) = 1 \otimes 1 \otimes 1 \otimes \Delta(a) \otimes 1$, we have $\varphi(r) = \varphi(s)$, where

$$s(x, y, z, u, v, w) = \Phi(x * y, z, w)\Phi(x, y, z)^{-1}$$
$$\times \Phi(u, v, w)^{-1}\Phi(x, u, v * w).$$

From (1.8) it follows that

$$s(x, y, z, u, v, w) = \Phi(x, y, z * w)^{-1}\Phi(y, z, w)\Phi(x, y * z, w)$$
$$\times \Phi(x, u * v, w)\Phi(x, u, v)\Phi(x * u, v, w)^{-1}.$$

From (1.30) and the analogous formula $\varphi(\Delta^{36}(a) \cdot h) = \varepsilon(a)h = \varphi(h \cdot \Delta^{14}(a))$ it follows that $\varphi(s) = \varphi(t)$, where $t(x, y, z, u, v, w) = \Phi(y, z, w)\Phi(x, u, v)$. From (1.18) it follows that $\varphi(t) = 1$. This proves (1.28).

If $(A, \Delta, \varepsilon, \Phi)$ is replaced by $(A, \Delta', \varepsilon, (\Phi^{321})^{-1})$, then γ and δ are replaced by $(S^{-1} \otimes S^{-1})(\gamma)$ and $(S^{-1} \otimes S^{-1})(\delta)$ (this follows easily from part 1 of the lemma), and S by S^{-1}. Therefore (1.29) follows from (1.28). ●

LEMMA 2. *Suppose given a k-algebra B, a homomorphism $f: A \to B$, an antihomomorphism $g: A \to B$, and elements $\rho, \sigma \in B$ such that*

$$\sum_i g(b_i)\rho f(c_i) = \varepsilon(a)\rho, \qquad \sum_j f(b_i)\sigma g(c_i) = \varepsilon(a)\sigma \tag{1.31}$$

for $a \in A$ and $\sum_i b_i \otimes c_i = \Delta(a)$, and

$$\sum_i f(X_i)\sigma g(Y_i)\rho f(Z_i) = 1 \tag{1.32}$$

where $\sum_i X_i \otimes Y_i \otimes Z_i = \Phi$,

$$\sum_j g(P_j)\rho f(Q_j)\sigma g(R_j) = 1, \tag{1.33}$$

where $\sum_j P_j \otimes Q_j \otimes R_j = \Phi^{-1}$. In addition, suppose given $\overline{\rho}, \overline{\sigma} \in B$ and an antihomomorphism $\overline{g}: A \to B$ also satisfying (1.31)–(1.33). Then there exists exactly one invertible element $F \in B$ such that $\overline{\rho} = F\rho$, $\overline{\sigma} = \sigma F^{-1}$, and $\overline{g}(a) = Fg(a)F^{-1}$ for $a \in A$. Furthermore, $F = \sum_i \overline{g}(P_i)\overline{\rho} f(Q_i)\sigma g(R_i)$ and $F^{-1} = \sum_i g(P_i)\rho f(Q_i)\overline{\sigma}\overline{g}(R_i)$.

The proof is similar to that of Proposition 1.1, which considered the case $B = A$, $f = \mathrm{id}$. •

Applying Lemma 2 to $B = A \otimes A$, $f = \Delta$, $g(a) = \Delta(S(a))$, $\rho = \Delta(\alpha)$, $\sigma = \Delta(\beta)$, $\overline{g}(a) = (S \otimes S)(\Delta'(a))$, $\overline{\rho} = \gamma$, and $\overline{\sigma} = \delta$, we obtain an invertible element $F \in A \otimes A$ such that

$$F\Delta(S(a))F^{-1} = (S \otimes S)(\Delta'(a)), \tag{1.34}$$

$$\gamma = F \cdot \Delta(\alpha). \tag{1.35}$$

In fact,

$$F = \sum_i (S \otimes S)(\Delta'(P_i)) \cdot \gamma \cdot \Delta(Q_i \beta S(R_i)), \tag{1.36}$$

$$F^{-1} = \sum_i \Delta(S(P_i)\alpha Q_i) \cdot \delta \cdot (S \otimes S)(\Delta'(R_i)).$$

From (1.34) it follows that $\tilde{\Delta}(a) = F\Delta(a)F^{-1}$. It remains to prove the equality (1.12). This can be written in the form

$$\begin{aligned}(S \otimes S \otimes S)(\Phi^{321}) \cdot (F \otimes 1) \cdot (\Delta \otimes \mathrm{id})(F) \\ = (1 \otimes F) \cdot (\mathrm{id} \otimes \Delta)(F) \cdot \Phi.\end{aligned} \tag{1.37}$$

From (1.36) and (1.34) it follows that

$$\begin{aligned}(F \otimes 1) \cdot (\Delta \otimes \mathrm{id})(F) &= (F \otimes 1) \cdot \sum_i (\Delta S \otimes S)(\Delta'(P_i)) \cdot (\Delta \otimes \mathrm{id})(\gamma) \\ &\quad \times (\Delta \otimes \mathrm{id})(\Delta(Q_i \beta S(R_i))) \\ &= \sum_i (S \otimes S \otimes S)(\Delta' \otimes \mathrm{id})(\Delta'(P_i)) \cdot (F \otimes 1) \\ &\quad \times (\Delta \otimes \mathrm{id})(\gamma) \cdot (\Delta \otimes \mathrm{id})(\Delta(Q_i \beta S(R_i))).\end{aligned}$$

Therefore

$$\begin{aligned}(S\otimes S\otimes S)(\Phi^{321})\cdot(F\otimes 1)\cdot(\Delta\otimes\mathrm{id})(\gamma)\\ =\sum_i(S\otimes S\otimes S)(\mathrm{id}\otimes\Delta')(\Delta'(P_i))(S\otimes S\otimes S)(\Phi^{321})\\ \times(F\otimes 1)\cdot(\Delta\otimes\mathrm{id})(\gamma)\cdot(\Delta\otimes\mathrm{id})(\Delta(Q_i\beta S(R_i))).\end{aligned}$$

Transforming similarly the right-hand side of (1.37), we reduce the proof of (1.37) to verification of the equality

$$(S\otimes S\otimes S)(\Phi^{321})\cdot(F\otimes 1)\cdot(\Delta\otimes\mathrm{id})(\gamma)=(1\otimes F)\cdot(\mathrm{id}\otimes\Delta)(\gamma)\cdot\Phi. \tag{1.38}$$

From (1.24), (1.34), (1.35) it follows that

$$\begin{aligned}(F\otimes 1)\cdot(\Delta\otimes\mathrm{id})(\gamma)&=\sum_i F\cdot\Delta(S(U_i))\cdot\Delta(\alpha)\cdot\Delta(V_i)\otimes S(T_i)\alpha W_i\\ &=\sum_{i,j}(S\otimes S)(\Delta'(U_i))(S(U_j)\otimes S(T_j))(\alpha\otimes\alpha)(V_j\otimes W_j)\\ &\quad\times\Delta(V_i)\otimes S(T_i)\alpha W_i.\end{aligned}$$

Hence the left-hand side of (1.38) is equal to $\varphi(r)$, where φ is the k-linear mapping $A^{\otimes 6}\to A^{\otimes 3}$ taking $a\otimes b\otimes c\otimes d\otimes e\otimes f$ into $S(c)\alpha d\otimes S(b)\alpha e\otimes S(a)\alpha f$, and

$$\begin{aligned}r=&\left(\sum_j 1\otimes T_j\otimes U_j\otimes V_j\otimes W_j\otimes 1\right)\\ &\times\sum_i(T_i\otimes\Delta(U_i)\otimes\Delta(V_i)\otimes W_i)\cdot(\Phi\otimes 1\otimes 1\otimes 1),\end{aligned}$$

or, in the other notation

$$\begin{aligned}&r(x,y,z,u,v,w)\\ &\quad=\Phi(z,u,v)^{-1}\Phi(y,z,u*v)\\ &\quad\quad\times\Phi(y*z,u*v,w)^{-1}\Phi(x,y*z,(u*v)*w)\Phi(x,y,z).\end{aligned}$$

Similarly the right-hand side of (1.38) is equal to $\varphi(s)$, where

$$\begin{aligned}s(x,y,z,u,v,w)=\Phi(y,v,w)^{-1}\cdot\Phi(x,y,v*w)\cdot\Phi(z,u,v*w)^{-1}\\ \times\Phi(x*y,z,u*(v*w))\Phi(u,v,w).\end{aligned}$$

Since $(\varphi((1\otimes 1\otimes\Delta(a)\otimes 1\otimes 1)h)=\varepsilon(a)\varphi(h)$ for $a\in A$ and $h\in A^{\otimes 6}$, to prove (1.38) it suffices to show that $r=\tilde{s}$, where

$$\begin{aligned}&\tilde{s}(x,y,z,u,v,w)\\ &\quad=\Phi(y,(z*u)*v,w)^{-1}\Phi(x,y,((z*u)*v)*w)\\ &\quad\quad\times\Phi(z*u,v,w)^{-1}\Phi(z,u,v*w)^{-1}\Phi(x*y,z,u*(v*w))\Phi(u,v,w).\end{aligned}$$

The equality $r=\tilde{s}$ can be deduced from Mac Lane's theorem [10] that the commutativity of (1.5) implies the commutativity of any diagram consisting of associativity morphisms. But here is a direct proof:

$$\begin{aligned}
r(x, y, z, u, v, w) &= \Phi(z, u, v)^{-1}\Phi(y, z, u*v)\Phi(y*z, u*v, w)^{-1} \\
&\quad \times \Phi(y, z, (u*v)*w)^{-1}\Phi(x, y, z*((u*v)*w)) \\
&\quad \times \Phi(x*y, z, (u*v)*w) \\
&= \Phi(z, u, v)^{-1}\Phi(y, z*(u*v), w)^{-1}\Phi(z, u*v, w)^{-1} \\
&\quad \times \Phi(x, y, z*((u*v)*w))\Phi(x*y, z, (u*v)*w) \\
&= \Phi(z, u, v)^{-1}\Phi(y, z*(u*v), w)^{-1} \\
&\quad \times \Phi(x, y, (z*(u*v))*w)\Phi(z, u*v, w)^{-1} \\
&\quad \times \Phi(x*y, z, (u*v)*w) \\
&= \Phi(y, (z*u)*v, w)^{-1}\Phi(x, y, ((z*u)*v)*w) \\
&\quad \times \Phi(z, u, v)^{-1}\Phi(z, u*v, w)^{-1}\Phi(x*y, z, (u*v)*w) \\
&= \Phi(y, (z*u)*v, w)^{-1}\Phi(x, y, ((z*u)*v)*w) \\
&\quad \times \Phi(z*u, v, w)^{-1}\Phi(z, u, v*w)^{-1}\Phi(u, v, w) \\
&\quad \times \Phi(x*y, z, (u*v)*w) \\
&= \tilde{s}(x, y, z, u, v, w). \quad \bullet
\end{aligned}$$

Propositions 1.3–1.5 below will be used to prove Theorem 1.6, which asserts that a deformation of a quasi-Hopf algebra as a quasibialgebra is quasi-Hopf.

Proposition 1.3. *Let $(A, \Delta, \varepsilon, \Phi)$ be a quasibialgebra, and suppose given an antiautomorphism S of A and elements $\alpha, \beta \in A$ that satisfy* (1.17). *Then the left-hand sides of* (1.18) *and* (1.19) *belong to the center of A; and if one of them is equal to* 1, *so is the other.*

Proof (cf. the proof of Proposition 1.1). Put $g = \sum_i X_i\beta S(Y_i)\alpha Z_i$ and $h = \sum_j S(P_j)\alpha Q_j\beta S(R_j)$, where we have $\sum_i X_i \otimes Y_i \otimes Z_i = \Phi$ and $\sum_j P_j \otimes Q_j \otimes R_j = \Phi^{-1}$. Applying to both sides of the equality $(\Delta \otimes \mathrm{id})(\Delta(a)) \cdot \Phi^{-1} = \Phi^{-1}(\mathrm{id} \otimes \Delta)(\Delta(a))$, $a \in A$, the k-linear mapping $A \otimes A \otimes A \to A$ taking $b \otimes c \otimes d$ into $S(b)\alpha c\beta S(d)$, we find that h is central. Applying to the equality $\Phi\cdot(\Delta\otimes\mathrm{id})(\Delta(a)) = (\mathrm{id}\otimes\Delta)(\Delta(a))\cdot\Phi$ we find that g is central. Applying to (1.21) the mapping $b\otimes c\otimes d\otimes e \mapsto S(b)\alpha c\beta S(d)\alpha e$, we find that $\alpha g = h\alpha$. From this and the centrality of g and h it follows that $fg = fh$ for $f \in A\alpha A$. Putting $f = g$ and $f = h$, we find that $g^2 = gh = h^2$. Therefore $g = 1 \Leftrightarrow h = 1$. $\bullet$

Proposition 1.4. *Let $(A, \Delta, \varepsilon, \Phi)$ be a quasibialgebra. Suppose there exist antiautomorphisms $S, \overline{S}$ of A and elements $\overline{\alpha}, \beta \in A$ such that:* 1) *if $a \in A$ and $\Delta(a) = \sum_i b_i \otimes c_i$, then $\sum_i \overline{S}(b_i)\overline{\alpha}c_i = \varepsilon(a)\overline{\alpha}$ and $\sum_i b_i\beta S(c_i) = \varepsilon(a)\beta$;* 2) *the element $u = \sum_i \overline{S}(P_i)\overline{\alpha}Q_i\beta S(R_i)$ is invertible, where $\sum_i P_i\otimes Q_i\otimes R_i = \Phi^{-1}$. Then $(A, \Delta, \varepsilon, \Phi)$ is a quasi-Hopf algebra.*

Proof. Arguing as in the proof of Proposition 1.1, we find that $uS(a) = \overline{S}(a)u$ for $a \in A$, and therefore $S(a)u^{-1} = u^{-1}\overline{S}(a)$. Put $\alpha = u^{-1}\overline{\alpha}$. Then $\sum_i S(P_i)\alpha Q_i\beta S(R_i) = 1$; while if $a \in A$ and $\Delta(a) = \sum_i b_i \otimes c_i$, then $\sum_i S(b_i)\alpha c_i = \varepsilon(a)\alpha$. Thus, S, α, β satisfy (1.17) and (1.19), and so also (1.18) (see Proposition 1.3). $\bullet$

PROPOSITION 1.5. *Let* $(A, \Delta, \varepsilon, \Phi)$ *be a quasi-Hopf algebra, and suppose* S, α, β *satisfy* (1.17)–(1.19). *Define* $w, \tilde{w} \in A \otimes A$ *by the formulas* $w = \sum_j S(P_j)\alpha Q_j \otimes R_j$ *and* $\tilde{w} = \sum_i Y_i S^{-1}(\beta) S^{-1}(X_i) \otimes Z_i$, *where* $\sum_j P_j \otimes Q_j \otimes R_j = \Phi^{-1}$ *and* $\sum_i X_i \otimes Y_i \otimes Z_i = \Phi$. *Denote by* J_l *and* J_r *the left and right ideals of* $A \otimes A$ *spanned by* $\Delta(\operatorname{Ker} \varepsilon)$. *Then*: 1) *the* k*-linear mappings* $\varphi, \tilde{\varphi}: A \otimes A \to A \otimes A$ *given by* $\varphi(a \otimes b) = (a \otimes 1) w \Delta(b)$ *and* $\tilde{\varphi}(a \otimes b) = \Delta(b) \tilde{w} (a \otimes 1)$ *are bijective*; 2a) *the mapping* $a \otimes b \mapsto (\mathrm{id} \otimes \varepsilon)(\varphi^{-1}(a \otimes b))$ *induces a bijection* $(A \otimes A)/J_l \to A$; 2b) $(\mathrm{id} \otimes \varepsilon)(\varphi^{-1}(a \otimes b)) = a\beta S(b)$; 3a) *the mapping* $a \otimes b \mapsto (\mathrm{id} \otimes \varepsilon)(\tilde{\varphi}^{-1}(a \otimes b))$ *induces a bijection* $(A \otimes A)/J_r \to A$; 3b) $(\mathrm{id} \otimes \varepsilon)(\tilde{\varphi}^{-1}(a \otimes b)) = S^{-1}(b) S^{-1}(\alpha) a$.

PROOF. Parts 2a) and 3a) follow from 1); while 1), 2b) and 3b) are easily derived from the formulas

$$(S \otimes \mathrm{id})\tilde{\varphi}(S^{-1} \otimes \mathrm{id})\varphi = \mathrm{id}, \tag{1.39}$$

$$(S^{-1} \otimes \mathrm{id})\varphi(S \otimes \mathrm{id})\tilde{\varphi} = \mathrm{id}. \tag{1.40}$$

(1.40) follows from (1.39), since if the multiplication in A is replaced by the opposite operation, then S is replaced by S^{-1} and φ by $\tilde{\varphi}$ (see Remark 4 to the definition of quasi-Hopf algebra). What remains is to prove (1.39). It is easily verified that $((S \otimes \mathrm{id})\tilde{\varphi}(S^{-1} \otimes \mathrm{id}))(u) = ((\mathrm{id} \otimes \Delta)(u) \cdot \Phi) \circ (\beta \otimes 1)$, where $u \in A \otimes A$ and $\circ$ means the following $(A \otimes A \otimes A)$-module structure on $A \otimes A$: $(a \otimes b \otimes c) \circ (d \otimes e) = adS(b) \otimes ce$. Therefore

$$\begin{aligned}
&(S \otimes \mathrm{id})\tilde{\varphi}(S^{-1} \otimes \mathrm{id})\varphi(a \otimes b) \\
&\quad = ((a \otimes 1 \otimes 1) \cdot (\mathrm{id} \otimes \Delta)(w) \cdot (\mathrm{id} \otimes \Delta)(\Delta(b)) \cdot \Phi) \circ (\beta \otimes 1) \\
&\quad = ((a \otimes 1 \otimes 1) \cdot (\mathrm{id} \otimes \Delta)(w) \cdot \Phi \cdot (\Delta \otimes \mathrm{id})(\Delta(b))) \circ (\beta \otimes 1) \\
&\quad = ((a \otimes 1 \otimes 1) \cdot (\mathrm{id} \otimes \Delta)(w) \cdot \Phi) \circ (\beta \otimes b) \\
&\quad = (a \otimes 1) v \cdot (1 \otimes b),
\end{aligned}$$

where $v = ((\mathrm{id} \otimes \Delta)(w) \cdot \Phi) \circ (\beta \otimes 1)$. We have $v = \psi((\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi^{-1}) \cdot (1 \otimes \Phi))$, where $\psi: A \otimes A \otimes A \otimes A \to A \otimes A$ takes $c \otimes d \otimes e \otimes f$ into $S(c)\alpha d \beta S(e) \otimes f$. Since

$$(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi^{-1}) \cdot (1 \otimes \Phi) = (\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\Phi) \cdot (\Phi^{-1} \otimes 1) \cdot (\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Phi^{-1}),$$

it follows that $v = 1$. ●

THEOREM 1.6. *Let* A *be a quasialgebra over* k *that is flat as a* k*-module, and* I *a nilpotent ideal in* k. *Suppose* A/IA *is a quasi-Hopf algebra over* k/I. *Then* A *is a quasi-Hopf algebra over* k.

PROOF. Put $B = A/IA$. The antiautomorphism of B that figures in the definition of quasi-Hopf algebra will be denoted by S_B; and similarly for the notation $\alpha_B, \beta_B \in B$. Define $w_B, \tilde{w}_B \in B \otimes B$ as in Proposition 1.5, and lift $w_B, \tilde{w}_B$ to $v, \tilde{v} \in A \otimes A$. Then the k-linear mappings $\psi, \tilde{\psi}: A \otimes A \to A \otimes A$ given by $\psi(a \otimes b) = (a \otimes 1)v\Delta(b)$, $\tilde{\psi}(a \otimes b) = \Delta(b)\tilde{v}(a \otimes 1)$ are bijective. This is a consequence of Proposition 1.5 and the following lemma.

LEMMA. *Let* $f: M \to N$ *be a morphism of flat* k*-modules that induces an isomorphism* $\overline{f}: M/IM \to N/IN$. *Then* f *is bijective.*

PROOF. That f is surjective is obvious. Since N is flat, we have $(\operatorname{Ker} f)/I \cdot (\operatorname{Ker} f) = \operatorname{Ker} \overline{f} = 0$; and therefore $\operatorname{Ker} f = 0$. ●

The mapping $a\otimes b \mapsto (\mathrm{id}\otimes\varepsilon)(\psi^{-1}(a\otimes b))$ induces a bijection $g: (A\otimes A)/J_l \to A$, where J_l has the same meaning as in Proposition 1.5. Using g, carry over to A the $(A\otimes A)$-module structure that is present on $(A\otimes A)/J_l$. Since the element $a\otimes 1 \in A\otimes A$ acts on A as left multiplication by a, the element $1\otimes b \in A\otimes A$ acts on A as right multiplication by $S(b)$, where S is some antihomomorphism $A \to A$. Since $S\,(\operatorname{mod} IA) = S_B$ (see part 2b) of Proposition 1.5), it follows from the lemma that S is bijective. Put $\beta = g(1)$. Then $\sum_i b_i\beta S(c_i) = \varepsilon(a)\beta$ for $a \in A$ and $\sum_i b_i \otimes c_i = \Delta(a)$; while from part 2b) of Proposition 1.5 it follows that $\beta \;(\operatorname{mod} IA) = \beta_B$.

Similarly, using the bijectivity of $\tilde{\psi}$, we construct an antiautomorphism $\sigma: A \to A$ and an element $\gamma \in A$ such that $\sigma\,(\operatorname{mod} IA) = S_B^{-1}$, $\gamma\,(\operatorname{mod} IA) = S_B^{-1}(\alpha_B)$, and $\sum_i \sigma(c_i)\gamma b_i = \varepsilon(a)\gamma$ for $a \in A$ and $\sum_i b_i \otimes c_i = \Delta(a)$. Put $\overline{S} = \sigma^{-1}$, $\overline{\alpha} = \overline{S}(\gamma)$. Then $\overline{S}\,(\operatorname{mod} IA) = S_B$, $\overline{\alpha}\,(\operatorname{mod} IA) = \alpha_B$, and $\sum_i \overline{S}(b_i)\overline{\alpha}c_i = \varepsilon(a)\overline{\alpha}$ for $a \in A$ and $\sum_i b_i \otimes c_i = \Delta(a)$. Put $u = \sum_i \overline{S}(P_i)\overline{\alpha}Q_i\beta S(R_i)$, where $\sum_i P_i \otimes Q_i \otimes R_i = \Phi^{-1}$. Since $u \equiv 1\,(\operatorname{mod} IA)$, the element u is invertible. It remains to apply Proposition 1.4. ●

§2. Quasi-Lie bialgebras

In this section we assume, for simplicity, that k is a field of characteristic 0.

We recall (see §3 of [2]) that a Lie bialgebra over k is a Lie k-algebra $\mathfrak{g}$ provided with a 1-cocycle $\delta: \mathfrak{g} \to \mathfrak{g}\otimes\mathfrak{g}$ that defines the structure of a Lie coalgebra on $\mathfrak{g}$. This means that $\delta(\mathfrak{g}) \subset \bigwedge^2 \mathfrak{g}$ and there the co-Jacobi identity $\operatorname{Alt}(\delta\otimes\mathrm{id})\delta = 0$ is satisfied, where $\operatorname{Alt}: \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$ is the alternation. The term "1-cocycle" means that δ is linear over k and $\delta([x, y]) = [x\otimes 1 + 1\otimes x, \delta(y)] - [y\otimes 1 + 1\otimes y, \delta(x)]$.

A Lie bialgebra is the classical analogue of a Hopf algebra. To explain (in part) what this means, we make some definitions. By a Hopf QUE-algebra over $k[[h]]$ we mean a topological Hopf algebra A over $k[[h]]$ such that: 1) the Hopf algebra A/hA is a universal enveloping algebra; 2) as a topological $k[[h]]$-module, A is isomorphic to $V[[h]]$ for some vector space V over k (a base of neighborhoods of zero in $V[[h]]$ is given by $h^n V[[h]]$, $n \in \mathbb{N}$). Let us note that: a) QUE is an abbreviation for "quantized universal enveloping"; b) in [2], Hopf QUE-algebras are called simply QUE-algebras; c) the term "topological Hopf algebra" means, in particular, that the comultiplication Δ maps A into the *completion* $A\hat{\otimes}A$ of the tensor product; d) since $\operatorname{char} k = 0$, the Lie algebra $\mathfrak{g}$ over k such that $A/hA = U\mathfrak{g}$ is unique: $\mathfrak{g} = \{a \in A/hA \mid \Delta(a) = a\otimes 1 + 1\otimes a\}$. It turns out [2] that if A is a Hopf QUE-algebra with $A/hA = U\mathfrak{g}$, then $\mathfrak{g}$ has a Lie bialgebra structure: the cocommutator $\delta: \mathfrak{g} \to \mathfrak{g}\otimes\mathfrak{g}$ is given by $\delta(x) = h^{-1}(\Delta(a) - \Delta'(a)) \bmod h$, where a is an inverse image of x in A and $\Delta': A \to A\hat{\otimes}A$ is the opposite comultiplication. The Lie bialgebra $(\mathfrak{g}, \delta)$ is called the classical limit of A, and A the quantization of $(\mathfrak{g}, \delta)$.

We describe now the classical analogues of quasi-Hopf algebras.

DEFINITION. By a quasi-Lie bialgebra is meant a triple $(\mathfrak{g}, \delta, \varphi)$, where $\mathfrak{g}$ is a Lie algebra, δ a 1-cocycle $\mathfrak{g} \to \bigwedge^2 \mathfrak{g} \subset \mathfrak{g}\otimes\mathfrak{g}$, and $\varphi \in \bigwedge^3 \mathfrak{g} \subset \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$, with the following equalities satisfied:

$$\tfrac{1}{2}\mathrm{Alt}(\delta\otimes\mathrm{id})\delta(x) = [x\otimes 1\otimes 1+1\otimes x\otimes 1+1\otimes 1\otimes x\,,\,\varphi], \qquad x\in\mathfrak{g}, \tag{2.1}$$

$$\mathrm{Alt}(\delta\otimes\mathrm{id}\otimes\mathrm{id})(\varphi)=0. \tag{2.2}$$

Let us note that in defining the operator $\mathrm{Alt}\colon \mathfrak{g}^{\otimes n}\to\mathfrak{g}^{\otimes n}$ we do not include the factor $(n!)^{-1}$.

REMARKS. 1) Quasi-Lie and Lie bialgebras are not bialgebras in the sense of §1.

2) For any 1-cocycle $\delta\colon\mathfrak{g}\to\bigwedge^2\mathfrak{g}$ the mapping $\mathrm{Alt}\circ(\delta\otimes\mathrm{id})\circ\delta\colon\mathfrak{g}\to\bigwedge^3\mathfrak{g}$ is also a 1-cocycle. (2.1) says that $\frac{1}{2}\mathrm{Alt}\circ(\delta\otimes\mathrm{id})\circ\delta$ is the coboundary of φ.

3) Let e_i be a basis for $\mathfrak{g}$, $[e_i\,,e_j]=c_{ij}^k e_k$, $\delta(e_i)=f_i^{jk}e^j\otimes e^k$, and $\varphi=\varphi^{ijk}e_i\otimes e_j\otimes e_k$ (here and below, summation is implied for repeated indices). Then the axioms for a quasi-Lie bialgebra mean that

$$c_{ij}^k=-c_{ji}^k,\qquad f_i^{jk}=-f_i^{kj},\qquad \varphi^{ijk}=-\varphi^{jik}=-\varphi^{ikj}, \tag{2.3}$$

$$\mathop{\mathrm{Alt}}_{i,j,k} c_{ij}^r c_{rk}^l=0,\qquad c_{rs}^k f_k^{ij}=\mathop{\mathrm{Alt}}_{r,s}\mathop{\mathrm{Alt}}_{i,j} c_{ar}^i f_s^{ja}, \tag{2.4}$$

$$\mathop{\mathrm{Alt}}_{i,j,k}(f_r^{ij}f_l^{rk}-c_{lm}^k\varphi^{ijm})=0,\qquad \mathop{\mathrm{Alt}}_{i,j,k,l} f_r^{ij}\varphi^{rkl}=0. \tag{2.5}$$

Let $(\mathfrak{g}\,,\delta\,,\varphi)$ be a quasi-Lie bialgebra, and suppose $r\in\bigwedge^2\mathfrak{g}$. Put

$$\overline{\delta}(x)=\delta(x)+[x\otimes 1+1\otimes x\,,\,r], \tag{2.6}$$

$$\overline{\varphi}=\varphi+\tfrac{1}{2}\mathrm{Alt}(\delta\otimes\mathrm{id})r-\langle r\,,r\rangle, \tag{2.7}$$

where $\langle r\,,r\rangle=[r^{12}\,,r^{13}]+[r^{12}\,,r^{23}]+[r^{13}\,,r^{23}]$ is the left-hand side of the classical Yang-Baxter equation (see §4 of [2]). Then it is easily verified that $(\mathfrak{g}\,,\overline{\delta}\,,\overline{\varphi})$ is also a quasi-Lie bialgebra. We shall say that $(\mathfrak{g}\,,\overline{\delta}\,,\overline{\varphi})$ is obtained from $(\mathfrak{g}\,,\delta\,,\varphi)$ by *twisting via* r. Twisting via r_1+r_2 is equivalent to twisting first via r_1, then via r_2.

DEFINITION. By a quasi-Hopf QUE-algebra over $k[[h]]$ is meant a topological quasi-Hopf algebra $(A\,,\Delta\,,\varepsilon\,,\Phi)$ over $k[[h]]$ such that: 1) $\Phi\equiv 1 \bmod h$; 2) the Hopf algebra A/hA is a universal enveloping algebra; 3) as a topological $k[[h]]$-module, A is isomorphic to $V[[h]]$ for some vector space V over k; 4) $\mathrm{Alt}\,\Phi\equiv 0 \bmod h^2$. Twisting for quasi-Hopf QUE-algebras is defined by formulas (1.11), (1.12), where $F\equiv 1 \bmod h$ and $(\varepsilon\otimes\mathrm{id})(F)=(\mathrm{id}\otimes\varepsilon)(F)=1$.

It is easily seen that the condition $\mathrm{Alt}\,\Phi\equiv 0 \bmod h^2$ is preserved under a twist.

PROPOSITION 2.1. *Let* $(A\,,\Delta\,,\varepsilon\,,\Phi)$ *be a quasi-Hopf* QUE-*algebra, with* $A/hA=U\mathfrak{g}$. *Then*:

1) *By an appropriate twist, we can make* $\Phi\equiv 1 \bmod h^2$.

2) *If* $\Phi\equiv 1 \bmod h^2$, *then putting* $\varphi=h^{-2}\,\mathrm{Alt}\,\Phi \bmod h$ *and defining* $\delta\colon\mathfrak{g}\to\mathfrak{g}\otimes\mathfrak{g}$ *in the same way as in the case that* A *is a Hopf* QUE-*algebra, we obtain a quasi-Lie bialgebra structure on* $\mathfrak{g}$.

3) *If* $(A\,,\tilde{\Delta}\,,\varepsilon\,,\tilde{\Phi})$ *is obtained from* $(A\,,\Delta\,,\varepsilon\,,\Phi)$ *by twisting via* F, *and* $\tilde{\Phi}\equiv\Phi\equiv 1 \bmod h^2$, *then the quasi-Lie bialgebra* $(\mathfrak{g}\,,\tilde{\delta}\,,\tilde{\varphi})$ *corresponding to* $(A\,,\tilde{\Delta}\,,\varepsilon\,,\tilde{\Phi})$ *is obtained from* $(\mathfrak{g}\,,\delta\,,\varphi)$ *by twisting via* $r=-h^{-1}\mathrm{Alt}\,F \bmod h$.

To prove Proposition 2.1, we shall need some facts concerning ogy H^n of the complex

$$0 \to k \xrightarrow{d_0} U\mathfrak{g} \xrightarrow{d_1} U\mathfrak{g} \otimes U\mathfrak{g} \xrightarrow{d_2} U\mathfrak{g} \otimes U\mathfrak{g} \otimes U\mathfrak{g} \xrightarrow{d_3} \cdots , \tag{2.8}$$

where $d_n = d_n^0 - d_n^1 + \cdots + (-1)^{n+1} d_n^{n+1}$, $d_n^i(a_1 \otimes \cdots \otimes a_n) = a_1 \otimes \cdots \otimes a_{i-1} \otimes \Delta(a_i) \otimes a_{i+1} \otimes \cdots \otimes a_n$ for $1 \le i \le n$, and $d_n^0(x) = 1 \otimes x$, $d_n^{n+1}(x) = x \otimes 1$. We observe that H^n is also the cohomology of the complex

$$0 \to B^0 \xrightarrow{d_0} B^1 \xrightarrow{d_1} B^2 \xrightarrow{d_2} B^3 \xrightarrow{d_3} \cdots , \tag{2.8a}$$

where $B^n = \operatorname{Ker} s_n^1 \cap \cdots \cap \operatorname{Ker} s_n^n$, $s_n^i(a_1 \otimes \cdots \otimes a_n) = \varepsilon(a_i) a_1 \otimes \cdots \otimes a_{i-1} \otimes a_{i+1} \otimes \cdots \otimes a_n$ (this follows from the fact that the k-modules $(u\mathfrak{g})^{\otimes n}$ form a cosimplicial k-module, in which the d_n^i are the boundary operators and the s_n^i the degeneracy operators). Since $d_n(\mathfrak{g}^{\otimes n}) = 0$, we have a mapping $\mathfrak{g}^{\otimes n} \to H^n$. We denote its restriction to $\bigwedge^n \mathfrak{g}$ by μ.

Proposition 2.2. 1) *μ is an isomorphism.*

2) Alt: $(U\mathfrak{g})^{\otimes n} \to (U\mathfrak{g})^{\otimes n}$ *maps cocycles into* $\bigwedge^n \mathfrak{g}$, *and coboundaries into* 0. *This gives rise, therefore, to a mapping $H^n \to \bigwedge^n \mathfrak{g}$, equal to $n!\mu^{-1}$.*

Outline of the proof. The only nontrivial part is the surjectivity of μ. Since $U\mathfrak{g}$ is isomorphic as a coalgebra (see [15], Chapter II, §1, Proposition 9) to the symmetric algebra $\mathrm{Sym}^*\mathfrak{g}$ with the usual comultiplication, the proof of surjectivity reduces to the analogous assertion concerning the complex

$$0 \to k \to \mathrm{Sym}^*\mathfrak{g} \to \mathrm{Sym}^*\mathfrak{g} \otimes \mathrm{Sym}^*\mathfrak{g} \to \cdots , \tag{2.9}$$

which is well known (in essence, this is the basic lemma in the theory of formal groups over a ring of characteristic 0). It can be proved, for example, by observing that the mth homogeneous component of the complex (2.9) (with respect to the grading induced by the usual grading in $\mathrm{Sym}^*\mathfrak{g}$) is isomorphic to $\mathfrak{g}^{\otimes m} \otimes_{S_m} \operatorname{Ker}(C^*(I^m) \to C^*(\partial I^m))$, where S_m is the symmetric group; I is an interval, regarded as a simplicial set (see [16], Chapter II, §2.5); ∂I^m is the boundary of the cube I^m; and C^* denotes the cochain complex of a simplicial set. •

Proof of Proposition 2.1. Part 1) follows from Proposition 2.2. Let us prove 2). From Proposition 2.2 it follows that $\varphi \in \bigwedge^3 \mathfrak{g} \subset U\mathfrak{g} \otimes U\mathfrak{g} \otimes U\mathfrak{g}$. We show that $\delta(\mathfrak{g}) \subset \bigwedge^2 \mathfrak{g}$, so that δ is well defined as a mapping $\mathfrak{g} \to \bigwedge^2 \mathfrak{g}$. The formula $\delta(x) = h^{-1}(\Delta(a) - \Delta'(a)) \bmod h$, where a is an inverse image of x in A, defines δ as a mapping $U\mathfrak{g} \to \bigwedge^2(U\mathfrak{g})$. Since $\Phi \equiv 1 \bmod h^2$, Δ is coassociative $\bmod h^2$. Therefore $(\Delta \otimes \mathrm{id})\delta(x) = (\mathrm{id} \otimes \delta)\Delta(x) + \sigma_{23}(\delta \otimes \mathrm{id})\Delta(x)$, $x \in U\mathfrak{g}$, where σ_{23} switches the second and third factors in $U\mathfrak{g} \otimes U\mathfrak{g} \otimes U\mathfrak{g}$. In particular, if $x \in \mathfrak{g}$, then $(\Delta \otimes \mathrm{id})\delta(x) = 1 \otimes \delta(x) + \sigma_{23}(\delta(x) \otimes 1)$; i.e., $\delta(x) \in \mathfrak{g} \otimes U\mathfrak{g}$. Thus, $\delta(\mathfrak{g}) \subset (\mathfrak{g} \otimes U\mathfrak{g}) \cap \bigwedge^2(U\mathfrak{g}) = \bigwedge^2 \mathfrak{g}$. If now $x, y \in U\mathfrak{g}$, then $\delta(xy) = \delta(x)\Delta(y) + \Delta(x)\delta(y)$, whence $\delta([x, y]) = [\Delta(x), \delta(y)] - (\Delta(y), \delta(x)]$. Therefore $\delta \colon \mathfrak{g} \to \bigwedge^2 \mathfrak{g}$ is a 1-cocycle. Equality (2.1) follows from the equality $\frac{1}{2}\mathrm{Alt}(\delta \otimes \mathrm{id})\delta(\overline{a}) = h^{-2}\mathrm{Alt}((\Delta \otimes \mathrm{id})\Delta(a) - (\mathrm{id} \otimes \Delta)\Delta(a)) \bmod h$, where $a \in A$ and $\overline{a}$ is the image of a in $U\mathfrak{g}$. Finally, to prove (2.2), it suffices to alternate with respect to x, y, z, u the equality (that follows from (1.8))

$$\psi(x, y, z * u) + \psi(x * y, z, u) = \psi(y, z, u) + \psi(x, y * z, u) + \psi(x, y, z),$$

where $\psi = h^{-2}(\Phi - 1) \bmod h^2$ and $x, y, z, u, *$ have the same meaning as in (1.8).

We now prove 3). Let $v = h^{-1}(F - 1)$, and $\overline{v}$ be the residue class of $v \bmod h$. Then $r = -\mathrm{Alt}\,\overline{v}$. Since $\Phi \equiv \tilde{\Phi} \bmod h^2$, we have

$$v(y, z) + v(x, y * z) - v(x * y, z) - v(x, y) \equiv 0 \mod h, \tag{2.10}$$

i.e., $\overline{v}$ is a 2-cocycle of the complex (2.8). Hence from Proposition 2.2 it follows that $r \in \bigwedge^2 \mathfrak{g}$. Equality (2.6) is readily verified. Using (2.10), we find that

$$\begin{aligned}\tilde{\Phi}(x, y, z) - \Phi(x, y, z) &\equiv h\{v(y, z) + v(x, y * z) - v(x * y, z) - v(x, y)\} \\ &\quad + h^2\{v(y, z)v(x, y * z) - v(x, y)v(x * y, z)\} \mod h^3,\end{aligned}$$

so that

$$\begin{aligned}\tilde{\varphi}(x, y, z) - \varphi(x, y, z) &= \tfrac{1}{2} \operatorname*{Alt}_{x, y, z} \{h^{-1} \times [v(x, y * z) - v(x, z * y)] \\ &\qquad - h^{-1}[v(x * y, z) - v(y * x, z)]\} \bmod h \\ &\qquad + \operatorname*{Alt}_{x, y, z} \{\overline{v}(y, z)\overline{v}(x, y * z) - \overline{v}(x, y)\overline{v}(x * y, z)\}.\end{aligned}$$

From this it is easy to arrive at (2.7). ●

In the situation of Proposition 2.1, we shall call $(\mathfrak{g}, \delta, \varphi)$ the *classical limit* of $(A, \Delta, \varepsilon, \Phi)$, and $(A, \Delta, \varepsilon, \Phi)$ the *quantization* of $(\mathfrak{g}, \delta, \varphi)$.

Let us review now the one-to-one correspondence between Lie bialgebras and Manin triples [2]. For simplicity we restrict ourselves first to the finite-dimensional case. By a Manin triple is meant a set $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$, where $\mathfrak{p}$ is a metrized Lie algebra (i.e., a Lie algebra on which a nondegenerate invariant symmetric bilinear form is given), and $\mathfrak{p}_1, \mathfrak{p}_2 \subset \mathfrak{p}$ are isotropic Lie subalgebras such that $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$. In this situation $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are Lagrangian subspaces of $\mathfrak{p}$ (i.e., $\mathfrak{p}_i^{\perp} = \mathfrak{p}_i$), and $\mathfrak{p}_2$ is canonically isomorphic to $\mathfrak{p}_1^*$. If $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ is a Manin triple, and $\delta: \mathfrak{p}_1 \to \mathfrak{p}_1 \otimes \mathfrak{p}_1$ is the mapping dual to the commutator mapping $\mathfrak{p}_2 \otimes \mathfrak{p}_2 \to \mathfrak{p}_2$, then $(\mathfrak{p}_1, \delta)$ is a Lie bialgebra. Conversely, if $(\mathfrak{g}, \delta)$ is a Lie bialgebra, put $\mathfrak{p} = \mathfrak{g} \oplus \mathfrak{g}^*$, $\mathfrak{p}_1 = \mathfrak{g}$, $\mathfrak{p}_2 = \mathfrak{g}^*$, endow $\mathfrak{p}$ with the natural scalar product, and take $\delta^*: \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ as the commutator in $\mathfrak{g}^*$. Then $[x, l]$ can be uniquely defined for $x \in \mathfrak{g}$, $l \in \mathfrak{g}^*$ so that $\mathfrak{p}$ is a Lie algebra and the scalar product in $\mathfrak{p}$ is invariant. If e_i is a basis in $\mathfrak{g}$, e^i the dual basis in $\mathfrak{g}^*$, $[e_i, e_j] = c_{ij}^k e_k$ and $\delta(e_i) = f_i^{jk} e^j \otimes e^k$, then the e_i and e^i form a basis in $\mathfrak{p}$, and

$$(e_i, e^j) = \delta_i^j, \qquad (e_i, e_j) = 0, \qquad (e^i, e^j) = 0, \tag{2.11}$$

$$[e^i, e^j] = f_k^{ij} e^k, \tag{2.12}$$

$$[e_i, e^j] = f_i^{jk} e_k + c_{li}^j e^l. \tag{2.13}$$

Now suppose $\mathfrak{p}$ and $\mathfrak{p}_1$ have the same meaning as above, and $\mathfrak{p}_2 \subset \mathfrak{p}$ is a Lagrangian subspace (but not, in general, a subalgebra) such that $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$. The commutator $\mathfrak{p}_2 \otimes \mathfrak{p}_2 \to \mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$ has two components: $\mathfrak{p}_2 \otimes \mathfrak{p}_2 \to \mathfrak{p}_2$ and $\mathfrak{p}_2 \otimes \mathfrak{p}_2 \to \mathfrak{p}_1$. The first defines, as before, a mapping $\delta: \mathfrak{p}_1 \to \mathfrak{p}_1 \otimes \mathfrak{p}_1$, the second an element $\psi \in \mathfrak{p}_1 \otimes \mathfrak{p}_1 \otimes \mathfrak{p}_1$. Put $\varphi = -\psi$. It is easily verified that: 1) $(\mathfrak{p}_1, \delta, \varphi)$ is a quasi-Lie bialgebra; 2) the correspondence so constructed between triples $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ of this form and quasi-Lie bialgebras is one-to-one; 3) changing $\mathfrak{p}_2$, for fixed $\mathfrak{p}$ and $\mathfrak{p}_1$, is equivalent to twisting the corresponding quasi-Lie bialgebra; 4) in the quasi-Lie case, (2.11) and (2.13) remain in force, while (2.12) is replaced by the formula

$$[e^i, e^j] = f_k^{ij} e^k - \varphi^{ijl} e_l, \tag{2.14}$$

where f_k^{ij} and φ^{ijl} have the same meanings as in (2.3)–(2.5). Since a Lagrangian subspace always has a Lagrangian complement, specifying a quasi-Lie bialgebra up to a twist is equivalent to specifying a *Manin pair*, i.e., a pair $(\mathfrak{p}, \mathfrak{g})$, where $\mathfrak{p}$ is a metrized Lie algebra and $\mathfrak{g}$ a Lagrangian subalgebra.

Up to now all Lie algebras have been assumed to be finite-dimensional. If $\dim \mathfrak{g} = \infty$, it is natural to regard $\mathfrak{g}^*$ as a topological vector space in which a neighborhood base for zero is formed by the $W^\perp$, where $W \subset \mathfrak{g}$ runs through the finite-dimensional subspaces. The space $\mathfrak{g}^*$ is linearly compact (i.e., $\mathfrak{g}^*$ is the projective limit of finite-dimensional discrete spaces), and $\mathfrak{g} \oplus \mathfrak{g}^*$ is locally linearly compact (i.e., has a linearly compact open subspace). It is easily shown that if $\mathfrak{p}$ is assumed locally linearly compact, $\mathfrak{p}_1$ discrete, and $\mathfrak{p}_2$ open, while the nondegeneracy of the scalar product in $\mathfrak{p}$ is understood in the topological sense ($\mathfrak{p} \to \mathfrak{p}^*$ is a topological isomorphism), then the above assertions concerning the connection between quasi-Lie (resp. Lie) bialgebras and Manin pairs (resp. triples) remain valid in the infinite-dimensional case.

EXAMPLE. Let X be a compact connected Riemann surface, E the field of meromorphic functions on X, and A the adèle ring of E. Consider a fixed absolutely simple finite-dimensional Lie algebra $\mathfrak{g}$ over E and a meromorphic differential ω on X, $\omega \neq 0$. Put $\mathfrak{p} = \mathfrak{g} \otimes_E A$. Regard $\mathfrak{g}$ and $\mathfrak{p}$ as algebras over $\mathbb{C}$, and define the scalar product $(u, v) = \sum_{x \in X} \operatorname{res}_x \{\omega \cdot \operatorname{Tr} \rho(u_x) \rho(v_x)\}$ in $\mathfrak{p}$, where $u, v \in \mathfrak{p} = \mathfrak{g} \otimes_E A$, $\rho: \mathfrak{g} \to \mathfrak{gl}(n, E)$ is a faithful representation, and u_x is the x-component of u. Then $(\mathfrak{p}, \mathfrak{g})$ is a Manin pair. As explained in §4 of [2], it is impossible to enlarge this pair to a Manin triple if the genus of X is greater than 1 (this is connected with the fact that Manin triples of the type in question correspond to nondegenerate solutions of the classical Yang-Baxter equation).

The class of Manin pairs obtained in this fashion can be extended somewhat by observing that if $(\mathfrak{p}, \mathfrak{g})$ is a Manin pair, and $\mathfrak{c}$ an open subalgebra in $\mathfrak{p}$ such that $\mathfrak{c} \supset \mathfrak{c}^\perp$, then putting $\mathfrak{p}' = \mathfrak{c}/\mathfrak{c}^\perp$, $\mathfrak{g}' = ((\mathfrak{g} \cap \mathfrak{c}) + \mathfrak{c}^\perp)/\mathfrak{c}^\perp$, we obtain again a Manin pair. In the example above we can put, e.g., $\mathfrak{c} = (\mathfrak{g} \otimes_E A_S) \times L$, where L is an open Lagrangian subalgebra in $\mathfrak{g} \otimes_E A^S$ (here $S \subset X$, $S \neq \varnothing$, A^S is the ring of adèles without S-components, and $A_S = A^{X/S}$). Then $\mathfrak{p}' = \mathfrak{g} \otimes_E A_S$, $\mathfrak{g}' \simeq \mathfrak{g} \cap \mathfrak{c}$. Of course, we must still take care of the existence of L. For this it suffices that $S \supset S_1 \cup S_2$, where S_1 is the set of zeros and poles of ω, and S_2 the set of branch points of the smallest extension $E' \supset E$ for which the algebra $\mathfrak{g} \otimes_E E'$ splits. Indeed, in this case there exists an isomorphism

$f\colon \mathfrak{g}\otimes_E A^S \xrightarrow{\sim} \mathfrak{a}\otimes_{\mathbb{C}} A^S$, where $\mathfrak{a}$ is a finite-dimensional simple Lie algebra over $\mathbb{C}$, and we can put $L = f^{-1}(\mathfrak{a}\otimes_{\mathbb{C}} O^S)$, where $O^S \subset A^S$ is the ring of integral adèles.

It would be desirable to study the question of quantization of the quasi-Lie bialgebras corresponding to Manin pairs of the above type. This class of quasi-Lie bialgebras contains, in particular (see [2], §§3, 4), algebras $\mathfrak{a}[u]$ (where $\mathfrak{a}$ is a simple Lie algebra over $\mathbb{C}$, $\dim\mathfrak{a} < \infty$) with a Lie bialgebra structure defined by rational solutions of the classical Yang-Baxter equation, and also affine Lie algebras with a Lie bialgebra structure defined by trigonometric solutions. For these bialgebras the quantization is known (see [2], §6).

§3. Quasitriangular, triangular, and coboundary quasi-Hopf algebras

We recall here the definitions of quasitriangular, triangular, and coboundary Hopf algebras given in §10 of [2]. In all three cases we are concerned with a pair (A, R), where A is a Hopf algebra and R an invertible element of $A\otimes A$ such that
$$\Delta'(a) = R\Delta(a)R^{-1}, \qquad a \in A, \tag{3.1}$$
where Δ' is the opposite comultiplication. In addition, R must have certain supplementary properties: in the quasitriangular case, it must satisfy the equalities
$$(\Delta\otimes\mathrm{id})(R) = R^{13}R^{23}, \qquad (\mathrm{id}\otimes\Delta)(R) = R^{13}R^{12}; \tag{3.2}$$
in the triangular case, together with (3.2) it must satisfy the "unitary condition" $R^{21} = R^{-1}$; and in the coboundary case, the equalities $R^{21} = R^{-1}$, $(\varepsilon\otimes\varepsilon)(R) = 1$ and
$$R^{12}\cdot(\Delta\otimes\mathrm{id})(R) = R^{23}\cdot(\mathrm{id}\otimes\Delta)(R). \tag{3.3}$$
Here we have used the following system of notation: if $R = \sum_i a_i\otimes b_i$, then $R^{12} = \sum_i a_i\otimes b_i\otimes 1$, $R^{13} = \sum_i a_i\otimes 1\otimes b_i$, $R^{23} = \sum_i 1\otimes a_i\otimes b_i$, and $R^{21} = \sum_i b_i\otimes a_i$. The terms "triangular" and "quasitriangular" are explained by the fact that in the quasitriangular case R satisfies (see below) the Yang-Baxter quantum equation
$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}, \tag{3.4}$$
also called the quantum triangle equation. The term "coboundary" is motivated by consideration of the classical limit (see [2], §§4, 10).

What can be said about the monoidal category A-mod for these types of Hopf algebras A? If M and N are modules over A, then (3.1) allows us to define an A-module isomorphism $c = c_{M,N}\colon M\otimes N \xrightarrow{\sim} N\otimes M$, natural in M and N: namely, $c_{M,N} = \sigma\circ R_{M,N}$, where $R_{M,N}$ is the image of R in $\mathrm{End}_k(M\otimes N)$, while $\sigma\colon M\otimes N\xrightarrow{\sim} N\otimes M$ is the usual k-module isomorphism. If $R^{21} = R^{-1}$, then c is involutory, i.e., $c_{N,M}\circ c_{M,N} = \mathrm{id}$. If (3.2) holds, we have commutative diagrams
$$\begin{array}{ccc} (M_1\otimes M_2)\otimes M_3 & \xrightarrow{\sim} M_3\otimes(M_1\otimes M_2) & \simeq (M_3\otimes M_1)\otimes M_2 \\ \wr\| & & \uparrow\wr \\ M_1\otimes(M_2\otimes M_3) & \xrightarrow{\sim} M_1\otimes(M_3\otimes M_2) & \simeq (M_1\otimes M_3)\otimes M_2 \end{array} \tag{3.5a}$$
$$\begin{array}{ccc} M_1\otimes(M_2\otimes M_3) & \xrightarrow{\sim} (M_2\otimes M_3)\otimes M_1 & \simeq M_2\otimes(M_3\otimes M_1) \\ \wr\| & & \uparrow\wr \\ (M_1\otimes M_2)\otimes M_3 & \xrightarrow{\sim} (M_2\otimes M_1)\otimes M_3 & \simeq M_2\otimes(M_1\otimes M_3), \end{array} \tag{3.5b}$$

where $\simeq$ and $\tilde{\to}$ denote respectively the morphisms of associativity and of commutativity for c. If (3.3) holds, we have commutative diagrams of the form

$$\begin{array}{ccccc} (M_1 \otimes M_2) \otimes M_3 & \tilde{\to} & M_3 \otimes (M_1 \otimes M_2) & \tilde{\to} & M_3 \otimes (M_2 \otimes M_1) \\ \wr\| & & & & \wr\| \\ M_1 \otimes (M_2 \otimes M_3) & \tilde{\to} & (M_2 \otimes M_3) \otimes M_1 & \tilde{\to} & (M_3 \otimes M_2) \otimes M_1 . \end{array} \tag{3.6}$$

If $(\varepsilon \otimes \varepsilon)(R) = 1$, then $c_{k,k} = \mathrm{id}$. Finally, if (3.4) holds, we have the commutative diagram

$$\begin{array}{ccccccc} M_1 \otimes (M_2 \otimes M_3) & \tilde{\to} & M_1 \otimes (M_3 \otimes M_2) & \simeq & (M_1 \otimes M_3) \otimes M_2 & \tilde{\to} & (M_3 \otimes M_1) \otimes M_2 \\ \wr\| & & & & & & \wr\| \\ (M_1 \otimes M_2) \otimes M_3 & & & & & & M_3 \otimes (M_1 \otimes M_2) \\ \downarrow\wr & & & & & & \downarrow\wr \\ (M_2 \otimes M_1) \otimes M_3 & & & & & & M_3 \otimes (M_2 \otimes M_1) \\ \wr\| & & & & & & \wr\| \\ M_2 \otimes (M_1 \otimes M_3) & \tilde{\to} & M_2 \otimes (M_3 \otimes M_1) & \simeq & (M_3 \otimes M_2) \otimes M_1 & \tilde{\to} & (M_3 \otimes M_2) \otimes M_1 \end{array} \tag{3.7}$$

Thus, if (A, R) is a triangular Hopf algebra, then the diagrams (3.5) are commutative and c is involutory; i.e., A-mod is a symmetric monoidal category [9], or, in the terminology of [14], a tensor category (we note that if c is involutory, then commutativity of (3.5a) is equivalent to that of (3.5b); and similarly, if $R^{21} = R^{-1}$, then the left-hand equality in (3.2) is equivalent to the right). A monoidal category together with a not necessarily involutory commutativity morphism such that (3.5a) and (3.5b) are commutative is called a quasitensor category. To this class belong the categories of modules over quasitriangular Hopf algebras. Finally, if (A, R) is a coboundary of Hopf algebra, then c is involutory, $c_{k,k} = \mathrm{id}$, and the diagrams of the form (3.6) are commutative. Such categories are called *coboundary*.

Let us make use now of the notational system of (1.7)–(1.10). Equality (3.1) translates into

$$a(y * x) = R(x, y)a(x * y)R(x, y)^{-1}, \qquad a \in A. \tag{3.8}$$

DEFINITION. By a quasitriangular quasi-Hopf algebra is meant a set $(A, \Delta, \varepsilon, \Phi, R)$, where $(A, \Delta, \varepsilon, \Phi)$ is a quasi-Hopf algebra, while R is an invertible element of $A \otimes A$ satisfying (3.8) and the equalities

$$\begin{aligned} R(x * y, z) = \Phi(z, x, y)R(x, z)\Phi(x, z, y)^{-1} \\ \times R(y, z)\Phi(x, y, z), \end{aligned} \tag{3.9a}$$

$$\begin{aligned} R(x, y * z) = \Phi(y, z, x)^{-1}R(x, z)\Phi(y, x, z) \\ \times R(x, y)\Phi(x, y, z)^{-1}. \end{aligned} \tag{3.9b}$$

Coboundary quasi-Hopf algebras are defined in the same way, but with (3.9a) and (3.9b) replaced by

$$R(x, y)R(x * y, z) = \Phi(z, y, x)R(y, z)R(x, y * z)\Phi(x, y, z) \tag{3.10}$$

together with the relations $R(y, x) = R(x, y)^{-1}$, $R(0, 0) = 1$. A quasitriangular quasi-Hopf algebra is called triangular if $R(y, x) = R(x, y)^{-1}$. For all three cases, a twist is defined by formulas (1.11), (1.12) and

$$\tilde{R}(x, y) = F(y, x)R(x, y)F(x, y)^{-1}, \tag{3.11}$$

where F is an invertible element of $A\otimes A$ such that $F(x, 0)=1=F(0, y)$.

It is easily verified that the category of modules over a quasitriangular (resp. triangular, coboundary) quasi-Hopf algebra is a quasitensor (resp. tensor, coboundary) category if the commutativity morphism is defined in the same way as for the Hopf case. It is easily verified also that the twist is well defined (i.e., that $(A, \tilde{\Delta}, \varepsilon, \tilde{\Phi}, \tilde{R})$ satisfies the same asioms as $(A, \Delta, \varepsilon, \Phi, R))$ and that the equivalence constructed in §1 between the monoidal categories C and $\tilde{C}$ corresponding to $(A, \Delta, \varepsilon, \Phi)$ and $(A, \tilde{\Delta}, \varepsilon, \tilde{\Phi})$ are compatible with the commutativity morphisms.

REMARKS. 1) If $R(y, x)=R(x, y)^{-1}$, then (3.9a) is equivalent to (3.9b).

2) For all three types of quasi-Hopf algebras, $R(x, 0)=1=R(0, y)$. To see this, it suffices in the quasitriangular case to put $y=0$ in (3.9), and in the coboundary case to put $y=0$ in (3.10) and use the equality $R(0, 0)=1$. Analogous proofs show that in quasitensor and coboundary categories the diagrams of the form

$$\begin{tikzcd}[column sep=small] & M \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\ k\otimes M \arrow[rr, "\sim"] & & M\otimes k \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & M \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\ k\otimes M & & M\otimes k \arrow[ll, "\sim"'] \end{tikzcd}$$

are commutative (the quasitensor case is considered in Theorem 8 of [11]).

3) If $(A, \Delta, \varepsilon, \Phi, R)$ is a quasitriangular quasi-Hopf algebra, then we have equality (3.10), as well as the equality

$$\begin{aligned} &R(x, y)\Phi(z, x, y)R(x, z)\Phi(x, z, y)^{-1}R(y, z)\Phi(x, y, z) \\ &\quad=\Phi(z, y, x)R(y, z)\Phi(y, z, x)^{-1}R(x, z)\Phi(y, x, z)R(x, y), \end{aligned} \tag{3.12}$$

generalizing (3.4) (in particular, a triangular quasi-Hopf algebra is coboundary). To prove (3.12), it suffices, using (3.9a), to write the left-hand side of (3.12) as $R(x, y)R(x*y, z)$, the right as $R(y*x, z)R(x, y)$, and apply (3.8). Equality (3.10) follows directly from (3.9) and (3.12).

4) The preceding remark has a category analogue: in a quasitensor category diagrams (3.6) and (3.7) are commutative.

For the rest of this section, it is assumed that k is a field of characteristic 0.

DEFINITION. By a quasitriangular quasi-Hopf QUE-algebra over $k[[h]]$ is meant a set $(A, \Delta, \varepsilon, \Phi, R)$, where $(A, \Delta, \varepsilon, \Phi)$ is a quasi-Hopf QUE-algebra (see §2), while $R\in A\hat{\otimes}A$ satisfies (3.8) and (3.9) and is congruent to 1 mod h.

Similar definitions are made for triangular and boundary quasi-Hopf QUE-algebras. The definition of twist in the QUE-case includes, of course, the condition $F\equiv 1 \bmod h$. We note that the congruence $\operatorname{Alt}\Phi\equiv 0 \bmod h^2$ that enters into the definition of quasi-Hopf QUE-algebra follows (assuming that $\Phi\equiv 1 \bmod h$, $R\equiv 1 \bmod h$) from the equalities (3.9) that enter into the definition of quasitriangularity. We note also that in the definition of coboundary quasi-Hopf QUE-algebra the condition $R(0, 0)=1$ need not be included, since it follows from the equality $R(0, 0)^2=1$, which in turn follows from the condition $R(y, x)=R(x, y)^{-1}$.

PROPOSITION 3.1. *Let* $(A, \Delta, \varepsilon, \Phi, R)$ *be a quasitriangular quasi-Hopf* QUE-*algebra over* $k[[h]]$, *and let* $A/hA=U\mathfrak{g}$. *Put* $t=h^{-1}(R^{21}R-1) \bmod h\in U\mathfrak{g}\otimes U\mathfrak{g}$. *Then*:

1) t *is a symmetric* $\mathfrak{g}$-*invariant element of* $\mathfrak{g}\otimes\mathfrak{g}$, *and remains unchanged under a twist of* $(A, \Delta, \varepsilon, \Phi, R)$.

2) *By an appropriate twist of* $(A, \Delta, \varepsilon, \Phi, R)$ *we can make* $h^{-1}(R-1) \bmod h = t/2$ *and* $\Phi \equiv 1 \bmod h^2$; *and under these conditions* $h^{-2}\operatorname{Alt}\Phi \bmod h = [t^{12}, t^{23}]/4$ *and* $\Delta' \equiv \Delta \bmod h^2$.

PROOF. 1) That t is symmetric is obvious. From (3.9b) it follows that

$$R(z, x*y)^{-1} = \Phi(z, x, y)R(z, x)^{-1}\Phi(x, z, y)^{-1} \times R(z, y)^{-1}\Phi(x, y, z). \tag{3.13}$$

From (3.9a) and (3.13) it follows that $t(x*y, z) = t(x, z) + t(y, z)$; i.e., $t \in \mathfrak{g} \otimes U\mathfrak{g}$. Thus, $t \in \operatorname{Sym}^2\mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{g}$. From (3.1) it follows that $[R^{21}R, \Delta(a)] = 0$ for $a \in A$. This implies $\mathfrak{g}$-invariance of t.

2) First we make $\Phi \equiv 1 \bmod h^2$ (see Proposition 2.1). Then from (3.9) it follows that $r \in \mathfrak{g} \otimes \mathfrak{g}$, where $r = h^{-1}(R-1) \bmod h$. Next, twisting via an element $F \in A\hat{\otimes}A$ such that $F \equiv 1 + h(r - r^{21})/2 \bmod h^2$ we can make $h^{-1}(R-1) \bmod h = t/2$ and still have $\Phi \equiv 1 \bmod h^2$. With these conditions (3.12) allows us to find $h^{-2}\operatorname{Alt}\Phi \bmod h$, and from (3.1) it follows that $\Delta' \equiv \Delta \bmod h^2$. ●

In the situation of Proposition 3.1 we shall call $(\mathfrak{g}, t)$ the *classical limit* of $(A, \Delta, \varepsilon, \Phi, R)$, and $(A, \Delta, \varepsilon, \Phi, R)$ the *quantization* of $(\mathfrak{g}, t)$. If $(A, \Delta, \varepsilon, \Phi, R)$ is a *triangular* quasi-Hopf QUE-algebra, then $t = 0$, so that the classical limit of $(A, \Delta, \varepsilon, \Phi, R)$ can naturally be said to be $\mathfrak{g}$ (without additional structure).

PROPOSITION 3.2. *Let* $(A, \Delta, \varepsilon, \Phi, R)$ *be a coboundary quasi-Hopf* QUE-*algebra over* $k[[h]]$, *and* $A/hA = U\mathfrak{g}$. *Then after a suitable twist,* $R \equiv 1 \bmod h^2$ *and* $\Phi \equiv 1 \bmod h^2$. *With these conditions,* $\Delta' \equiv \Delta \bmod h^2$, *while* $\varphi \overset{\text{det}}{=} h^{-2}\operatorname{Alt}\Phi \bmod h$ *is a* $\mathfrak{g}$-*invariant element of* $\bigwedge^3\mathfrak{g} \subset (U\mathfrak{g})^{\otimes 3}$ *independent of the arbitrariness in the choice of the twist.*

PROOF. First we make $\Phi \equiv 1 \bmod h^2$ (see Proposition 2.1). We then put $r = h^{-1}(R-1) \bmod h \in U\mathfrak{g} \otimes U\mathfrak{g}$ and observe that, by (3.10), r is a 2-cocycle of the complex (2.8). Since $R^{21} = R^{-1}$, we have $r^{21} = -r$, and by part 2) of Proposition 2.2, $r \in \bigwedge^2\mathfrak{g}$. Twisting now via an $F \in A\hat{\otimes}A$ such that $F \equiv 1 + hr/2 \bmod h^2$, we can make $R \equiv 1 \bmod h^2$ and still have $\Phi \equiv 1 \bmod h^2$. With these conditions satisfied, tat $\tilde{R}$ and $\tilde{\Phi}$ defined by formulas (3.11) and (1.12), the congruences $R \equiv 1 \bmod h^2$ and $\tilde{\Phi} \equiv 1 \bmod h^2$ mean that $h^{-1}(F-1) \bmod h$ is symmetric and is a 2-cocycle of the complex (2.8a). By Proposition 2.2, $h^{-1}(F-1) \bmod h$ is a coboundary; i.e., $F = F_1 \cdot (u^{-1} \otimes u^{-1}) \cdot \Delta(u)$, where $u \in A$, $F_1 \in A \otimes A$, $u \equiv 1 \bmod h$, and $F_1 \equiv 1 \bmod h^2$. If $F = (u^{-1} \otimes u^{-1})\Delta(u)$, then $\tilde{\Phi} = (u^{-1} \otimes u^{-1} \otimes u^{-1}) \cdot \Phi \cdot (u \otimes u \otimes u) \equiv \Phi \bmod h^3$; and if $F \equiv 1 \bmod h^2$, then $\operatorname{Alt}\tilde{\Phi} = \operatorname{Alt}\Phi \bmod h^3$. Thus, φ is independent of the arbitrariness in the choice of the twist. If $R \equiv 1 \bmod h^2$ and $\Phi \equiv 1 \bmod h^2$, then $\Delta' \equiv \Delta \bmod h^2$ in view of (3.1), so that in the quasi-Lie bialgebra $(\mathfrak{g}, \delta, \varphi)$ that is the classical limit of $(A, \Delta, \varepsilon, \Phi)$ the cocommutator δ is equal to 0. Hence by (2.1) φ is $\mathfrak{g}$-invariant. ●

In the situation of Proposition 3.2 we shall call $(\mathfrak{g}, \varphi)$ the *classical limit* of $(A, \Delta, \varepsilon, \Phi, R)$, and $(A, \Delta, \varepsilon, \Phi, R)$ the *quantization* of $(\mathfrak{g}, \varphi)$. In the triangular case, i.e., the case of equalities (3.9), we have $\varphi = 0$.

By the classical limit of a quasitriangular (resp. triangular, coboundary) Hopf QUE-algebra (A, R) is meant the pair $(\mathfrak{g}, r)$, where $\mathfrak{g}$ is the classical limit of A (see §2) and $r = h^{-1}(R-1) \bmod h$. It is easily seen that $(\mathfrak{g}, r)$ is a quasitriangular (resp. triangular, coboundary) Lie bialgebra in the sense of §4 of [2]; i.e., $\mathfrak{g}$ is a Lie bialgebra, $r \in \mathfrak{g} \otimes \mathfrak{g}$, the cocommutator $\delta: \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ is the coboundary of r, and, in addition, for the quasitriangular case $\langle r, r\rangle = 0$, for the coboundary case $r \in \bigwedge^2 \mathfrak{g}$, and for the triangular case $\langle r, r\rangle = 0$ and $r \in \bigwedge^2 \mathfrak{g}$. Here $\langle r, r\rangle = [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}]$. It is easily shown that if $(\mathfrak{g}, r)$ is the classical limit of a quasitriangular (resp. coboundary) Hopf QUE-algebra (A, R), then the classical limit of (A, R) as a quasitriangular (resp. coboundary) *quasi-Hopf* QUE-algebra is $(\mathfrak{g}, r + r^{21})$ (resp. $(\mathfrak{g}, \langle r, r\rangle)$).

REMARK. The "classical" objects considered here have natural interpretations in terms of Manin triples and pairs. For simplicity we restrict ourselves to the finite-dimensional case. Let $\mathfrak{g}$ be a Lie bialgebra, and $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ the corresponding Manin triple; i.e., $\mathfrak{p} = \mathfrak{g} \oplus \mathfrak{g}^*$, $\mathfrak{p}_1 = \mathfrak{g}$, $\mathfrak{p}_2 = \mathfrak{g}^*$. To an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ we associate the graph $\mathfrak{a}$ of the mapping $\mathfrak{g}^* \to \mathfrak{g}$ that takes $l \in \mathfrak{g}^*$ into $(\mathrm{id} \otimes l)(r)$. This gives a bijection $\mathfrak{g} \otimes \mathfrak{g} \to \{$subspaces $\mathfrak{a} \subset \mathfrak{p}$ such that $\mathfrak{p} = \mathfrak{p}_1 + \mathfrak{a}\}$. It is easily shown that $(\mathfrak{g}, r)$ is a quasitriangular (resp. coboundary, triangular) Lie bialgebra if and only if $\mathfrak{a}$ is an ideal (resp. $(\mathrm{ad}\, \mathfrak{p}_1)$ is an invariant Lagrangian subspace, Lagrangian ideal). Consider now sets $(\mathfrak{p}, \mathfrak{g}, \mathfrak{a})$, where $(\mathfrak{p}, \mathfrak{g})$ is a Manin pair, and $\mathfrak{a}$ a subspace of $\mathfrak{p}$ such that $\mathfrak{p} = \mathfrak{g} \oplus \mathfrak{a}$ and one of the following three conditions holds: 1) $\mathfrak{a}$ is an ideal; 2) $\mathfrak{a}$ is a Lagrangian subspace and $[\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{a}$; 3) $\mathfrak{a}$ is a Lagrangian ideal. It is easily verified that: in case 1), specifying $(\mathfrak{p}, \mathfrak{g}, \mathfrak{a})$ is equivalent to specifying a pair $(\mathfrak{g}, t)$, where t is a $\mathfrak{g}$-invariant element of $\mathrm{Sym}^2\mathfrak{g}$; in case 2), specifying $(\mathfrak{p}, \mathfrak{g}, \mathfrak{a})$ is equivalent to specifying a pair $(\mathfrak{g}, \varphi)$, where φ is a $\mathfrak{g}$-invariant element of $\bigwedge^3 \mathfrak{g}$; in case 3), specifying $(\mathfrak{p}, \mathfrak{g}, \mathfrak{a})$ is equivalent to specifying $\mathfrak{g}$ (without additional structure). Here t corresponds to the restriction to $\mathfrak{a} \simeq \mathfrak{g}^*$ of the scalar product in $\mathfrak{p}$, and φ to the restriction to $\mathfrak{a} \simeq \mathfrak{g}^*$ of the trilinear form $([x, z], y)$.

If $(\mathfrak{g}, r)$ is a quasitriangular Lie bialgebra, then $(\mathfrak{g}, (r - r^{21})/2)$ is a coboundary Lie bialgebra. Let us consider now the quantum analogue of this construction. Let $(A, \Delta, \varepsilon, \Phi)$ be a quasi-Hopf QUE-algebra over $k[[h]]$; suppose $R \in A \hat{\otimes} A$, $R \equiv 1 \bmod h$, and (3.1) holds. Put $\overline{R} = R \cdot (R^{21}R)^{-1/2}$. Then

$$\overline{R}^{21}\overline{R} = R^{21} \cdot (R \cdot R^{21})^{-1/2} \cdot R \cdot (R^{21} \cdot R)^{-1/2} = R^{21} \cdot (R \cdot R^{21})^{-1} \cdot R = 1;$$

i.e., $\overline{R}$ satisfies the "unitary condition". Since (3.1) implies that $[R^{21}R, \Delta(a)] = 0$ for $a \in A$, $\overline{R}$ also satisfies (3.1). The passage from R to $\overline{R}$ we call *unitarization*. Unitarization commutes with the twist (3.11).

PROPOSITION 3.3. *If R satisfies* (3.10), *so does* $\overline{R}$.

PROOF. It suffices to show that

$$\overline{R}(x, y)\overline{R}(x * y, z) = R(x, y)R(x * y, z) \times \{R(y, x)R(z, y * x)R(x, y)R(x * y, z)\}^{-1/2}, \quad (3.14a)$$

$$\overline{R}(y, z)\overline{R}(x, y * z) = R(y, z)R(x, y * z) \times \{R(z, y)R(z * y, x)R(y, z)R(x, y * z)\}^{-1/2}. \quad (3.14b)$$

Let us prove, e.g., (3.14a). Since $[a(x*y), R(y,x)R(x,y)] = 0$ for $a \in A$, we have

$$\begin{aligned}\overline{R}(x,y)\overline{R}(x*y,z) &= R(x,y)\cdot(R(y,x)R(x,y))^{-1/2} \\ &\quad \times R(x*y,z)\cdot(R(z,x*y)R(x*y,z))^{-1/2} \\ &= R(x,y)R(x*y,z) \\ &\quad \times \{R(y,x)R(x,y)R(z,x*y)R(x*y,z)\}^{-1/2} \\ &= R(x,y)R(x*y,z) \\ &\quad \times \{R(y,x)R(z,y*x)R(x,y)R(x*y,z)\}^{-1/2}. \quad \bullet\end{aligned}$$

From Proposition 3.3 it follows that the unitarization $(A, \Delta, \varepsilon, \Phi, \overline{R})$ of a quasitriangular quasi-Hopf QUE-algebra $(A, \Delta, \varepsilon, \Phi, R)$ is coboundary. It is not, in general, triangular. Indeed, if $(\mathfrak{g}, t)$ is the classical limit of $(A, \Delta, \varepsilon, \Phi, R)$, then from part 2) of Proposition 2.1 it follows that the classical limit of $(A, \Delta, \varepsilon, \Phi, \overline{R})$ is $(\mathfrak{g}, [t^{12}, t^{23}]/4)$. Hence if $[t^{12}, t^{23}] \neq 0$, then $(A, \Delta, \varepsilon, \Phi, \overline{R})$ is not triangular. In particular, this is the case for the unitarization of the quasitriangular Hopf QUE-algebra $U_h\mathfrak{g}$ (see [2], §§6, 13), where $\mathfrak{g}$ is a finite-dimensional simple Lie algebra.

Proposition 3.4. *If $(A, \Delta, \varepsilon, \Phi, R)$ is a coboundary quasi-Hopf* QUE-*algebra, then* (3.12) *implies* (3.9).

Proof. It suffices to use the following lemma, which applies not just to the QUE-case.

Lemma. *Let $u(x,y,z) = \{$right-hand side of (3.9a)$\}^{-1} \times \{$left-hand side of (3.9a)$\}$, and $v(x,y,z) = \{$right-hand side of (3.12)$\}^{-1} \times \{$left-hand side of (3.12)$\}$. Then $v = u^{-2}$.*

Proof. Since

$$\begin{aligned}R(x*y,z) &= \Phi(z,x,y)R(x,z)\Phi(x,z,y)^{-1} \\ &\quad \times R(y,z)\Phi(x,y,z)u(x,y,z), \quad R(x,y*z) \\ &= R(y*z,x)^{-1} \\ &= u(y,z,x)^{-1}\Phi(y,z,x)^{-1}R(x,z) \\ &\quad \times \Phi(y,x,z)R(x,y)\Phi(x,y,z)^{-1},\end{aligned}$$

equality (3.10) can be rewritten as

$$u(x,y,z) = b(z,x,y)b(y,z,x)u(y,z,x)^{-1}b(x,y,z),$$

where

$$b(x,y,z) = \Phi(y,z,x)^{-1}\cdot R(x,z)\Phi(y,x,z)R(x,y).$$

Iterating, we obtain

$$u(x,y,z) = b(z,x,y)b(y,z,x)b(x,y,z)u(x,y,z)^{-1};$$

i.e., $u^2 = v^{-1}$. $\bullet$

A simpler proof of Proposition 3.4 can be given by reducing it to the case $R = 1$ with the help of part 1) of the following proposition.

PROPOSITION 3.5. 1) *Any quasitriangular or coboundary quasi-Hopf* QUE-*algebra can by a suitable twist be brought into a symmetric form, i.e., such that* $R^{21} = R$. *In the coboundary case,* $R^{21} = R \Leftrightarrow R = 1$.

2) *Twisting via* F *preserves the symmetric form of a quasitriangular or coboundary quasi-Hopf* QUE-*algebra if and only if* $F^{21} = F$.

3) *Let* $(A, \Delta, \varepsilon, \Phi, R)$ *be a quasitriangular or coboundary quasi-Hopf* QUE-*algebra, with* $R^{21} = R$. *Then* $\Delta' = \Delta$, *and* Φ *satisfies the relation*

$$\Phi(z, y, x) = \Phi(x, y, z)^{-1}. \tag{3.15}$$

PROOF. The quation $\tilde{R}^{21} = \tilde{R}$, where $\tilde{R} = F^{21} R F^{-1}$, can be rewritten as $R^{21} R = (F^{-1} F^{21} R)^2$. Its general solution is therefore of the form $F = F_0 \cdot (R \cdot (R^{21} R)^{-1/2})^{1/2}$, where $F_0^{21} = F_0$. Let us now prove 3). From (3.1) it follows that $[R^{21} R, \Delta(a)] = 0$ for $a \in A$. Since $R^{21} = R$, we have $[R, \Delta(a)] = 0$; so by (3.1), $\Delta' = \Delta$. Now rewrite (3.10) as

$$\Phi(x, y, z) = \{R(y, z) R(x, y * z)\}^{-1} \Phi(z, y, x)^{-1} R(x, y) R(x * y, z).$$

Iterating this relation and using the equalities $R(y, x) = R(x, y)$, $\Delta' = \Delta$, we find that

$$\Phi(x, y, z) = \{R(y, z) R(x, y * z)\}^{-2} \Phi(x, y, z) \{R(x, y) R(x * y, z)\}^2.$$

Hence

$$\Phi(x, y, z) \{R(x, y) R(x * y, z)\}^2 \Phi(x, y, z)^{-1} = \{R(y, z) R(x, y * z)\}^2,$$

and therefore

$$\Phi(x, y, z) R(x, y) R(x * y, z) \Phi(x, y, z)^{-1} = R(y, z) R(x, y * z).$$

From this last equality and (3.10) we obtain (3.15). ●

REMARK. Equality (3.15) could have been proved in a different way by observing that in the coboundary case it follows immediately from (3.10), since $R = 1$, while the quasitriangular case reduces to the coboundary by means of Proposition 3.3. Conversely, one can prove Proposition 3.3 by reducing it, using part 1) of Proposition 3.5, to the special case $R^{21} = R$ and observing that in this case it reduces to (3.15).

We shall now construct bijections between the isomorphism classes of triangular, quasitriangular, and coboundary quasi-Hopf QUE-algebras, up to twist, and the isomorphism classes of the corresponding classical objects. The author does not know whether there exist natural bijections of this type for quasi-Hopf QUE-algebras that fail to satisfy (3.1), or for Hopf QUE-algebras (including triangular, quasitriangular, and coboundary).

PROPOSITION 3.6. *Any triangular quasi-Hopf* QUE-*algebra over* $k[[h]]$ *can by a suitable twist be brought into the form* $R = 1$, $\Phi = 1$.

PROOF. We can suppose (see Proposition 3.5) that $R = 1$ from the very first, and therefore $\Phi(z, y, x) = \Phi(x, y, z)^{-1}$. We show that if $\Phi \equiv 1 \bmod h^n$, then there exists an $F \in A \hat{\otimes} A$ such that $F \equiv 1 \bmod h^n$, $(\varepsilon \otimes \mathrm{id})(F) = (\mathrm{id} \otimes \varepsilon)(F) = 1$, $\tilde{\Phi} \equiv 1 \bmod h^{n+1}$, and $\tilde{R} = 1$, where $\tilde{\Phi}$ and $\tilde{R}$ are defined by the formulas (1.12) and (3.11). The equality $\tilde{R} = 1$ is equivalent with F being symmetric, while the congruence $\tilde{\Phi} \equiv 1 \bmod h^{n+1}$ means that the coboundary of the element $h^{-n}(F-1) \bmod h \in B^2$ (see (2.8a)) is equal to α, where $\alpha = h^{-n}(1-\Phi) \bmod h \in B^3$. It remains to prove that α is the coboundary of a symmetric element of B^2. From (1.2) it follows that α is a cocycle, and, from (3.9), (3.15), and the equality $R = 1$, that α belongs to the kernel of Alt: $(U\mathfrak{g})^{\otimes 3} \to (U\mathfrak{g})^{\otimes 3}$. Hence by Proposition 2.2, α is the coboundary of some $\beta \in B^2 \subset U\mathfrak{g} \otimes U\mathfrak{g}$. By (3.15), the coboundary of the element $\beta^{21} \in B^2$ is also equal to α. Therefore, replacing β by $(\beta + \beta^{21})/2$, we arrive at the desired equality $\beta^{21} = \beta$. ●

A triangular quasi-Hopf QUE-algebra such that $R = 1$ and $\Phi = 1$ is the same thing as a cocommutative Hopf QUE-algebra.

Proposition 3.7. *A cocommutative Hopf* QUE-*algebra has the form* $U\mathfrak{g}$, *where* $\mathfrak{g}$ *is a Lie algebra over* $k[[h]]$ *that is isomorphic as a* $k[[h]]$-*module to* $V[[h]]$ *for some vector space* V *over* k.

The condition on $\mathfrak{g}$ in Proposition 3.7 means that $\mathfrak{g}$ is a deformation of the Lie algebra $\mathfrak{g}_0$ over k, where $\mathfrak{g}_0 = \mathfrak{g}/h\mathfrak{g}$. Such algebras $\mathfrak{g}$ will be called *deformation algebras.* For deformation algebras $\mathfrak{g}$ we mean by $U\mathfrak{g}$ not the algebraic universal envelope, but its h-adic completion.

Proof. Let A be a cocommutative Hopf QUE-algebra, $A/hA = U\mathfrak{g}_0$, and $\varphi: A \to A$ the composite of the comultiplication $\Delta: A \to A\hat{\otimes}A$. We have $\varphi(I) \subset I$, where $I = \operatorname{Ker}(A \xrightarrow{\varepsilon} k[[h]])$. The operator $\varphi \bmod h$ acts as multiplication by 2^n on the image of the canonical mapping $\operatorname{Sym}^n\mathfrak{g}_0 \to I/hI \subset U\mathfrak{g}_0$. Therefore I is the topological direct sum of φ-invariant $k[[h]]$-submodules W_n, $n = 1, 2, \dots$, such that $\varphi - 2^n \cdot \mathrm{id}$ acts in W_n topologically nilpotently. Put also $W_0 = k[[h]] \cdot 1 \subset A$. Then $A = W_0 \oplus I$, and $\varphi|_{W_0} = \mathrm{id}$. From the cocommutativity of Δ it follows that φ is a coalgebra homomorphism. Therefore $\Delta(W_1) \subset (W_0 \otimes W_1) \oplus (W_1 \otimes W_0)$; i.e., if $a \in W_1$, then $\Delta(a) = b \otimes 1 + 1 \otimes b$ for some $b \in W_1$. In fact, $b = (\varepsilon \otimes \mathrm{id})(\Delta(a)) = a$; i.e., $\Delta(a) = a \otimes 1 + 1 \otimes a$. Conversely, if $\Delta(a) = a \otimes 1 + 1 \otimes a$, then $\varphi(a) = 2a$, so that $a \in W_1$. Thus, $W_1 = \{a \in A | \Delta(a) = a \otimes 1 + 1 \otimes a\}$, and therefore W_1 is a Lie algebra. Since W_1 is a direct summand of A, it is a deformation algebra, and the natural homomorphism $UW_1 \to A$ is bijective, since its reduction $\bmod h$ is bijective. ●

Proposition 3.8. *Let* $A = U\mathfrak{g}$, *where* $\mathfrak{g}$ *is a deformation algebra over* $k[[h]]$. *Regard* A *as being a triangular quasi-Hopf* QUE-*algebra by defining* Δ *and* ε *in the usual fashion and putting* $R = 1$, $\Phi = 1$. *Suppose* $(A, \tilde{\Delta}, \varepsilon, \tilde{\Phi}, \tilde{R})$ *is obtained from* $(A, \Delta, \varepsilon, \Phi, R)$ *by a twist via* F. *Then the equalities* $\tilde{R} = 1$, $\tilde{\Phi} = 1$ *are equivalent to* F *being representable in the form*

$$F = (u \otimes u)\Delta(u)^{-1}, \qquad u \in A,\ u \equiv 1 \mod h,\ \varepsilon(u) = 1. \tag{3.16}$$

If there exists a u *satisfying* (3.16), *it is unique up to replacement by* ue^{hv}, $v \in \mathfrak{g}$. *If* (3.16) *holds, then* $\operatorname{Ad} u$ *maps* $\mathfrak{g}$ *isomorphically onto* $\tilde{g} = \{a \in A |$ $\tilde{\Delta}(a) = a \otimes 1 + 1 \otimes a\}$.

Proof. We show that if $\tilde{R} = 1$, $\tilde{\Phi} = 1$, and $F \equiv 1 \bmod h^n$, then there exists a $w \in A$ such that $w \equiv 1 \bmod h^n$, $\varepsilon(w) = 1$, and $(w^{-1} \otimes w^{-1}) \cdot F \cdot \Delta(w) \equiv 1$ $\bmod h^{n+1}$ (the rest is obvious). Indeed, the element $h^{-n}(F-1) \in B^2$ (see

(2.8a)) is symmetric and is a cocycle. By Proposition 2.2 it is the coboundary of some $\gamma \in B^1$. It remains to choose w so that $\varepsilon(w) = 1$ and $w \equiv 1 + h^n\gamma \bmod h^{n+1}$. ●

Two triangular quasi-Hopf QUE-algebras $(A, \Delta, \varepsilon, \Phi, R)$ and $(\overline{A}, \overline{\Delta}, \overline{\varepsilon}, \overline{\Phi}, \overline{R})$ are called equivalent if the set $(\overline{A}, \overline{\Delta}, \overline{\varepsilon}, \overline{\Phi}, \overline{R})$ is isomorphic to the set $(A, \tilde{\Delta}, \varepsilon, \tilde{\Phi}, \tilde{R})$ obtained from $(A, \Delta, \varepsilon, \Phi, R)$ by a twist. Propositions 3.6–3.8 show that the equivalence classes of triangular quasi-Hopf QUE-algebras are in one-to-one correspondence with the isomorphism classes of deformation algebras $\mathfrak{g}$. Let us note, however, that a triangular quasi-Hopf QUE-algebra corresponds, strictly speaking, not to a Lie algebra $\mathfrak{g}$, but to a class of such algebras, where the isomorphism between two algebras $\mathfrak{g}$ and $\overline{\mathfrak{g}}$ is defined not canonically, but up to automorphisms of the form $\exp(h \cdot \operatorname{ad} v)$, $v \in \mathfrak{g}$.

We describe now the category C of triangular quasi-Hopf QUE-algebras over $k[[h]]$. By a morphism $(A, \Delta, \varepsilon, \Phi, R) \to (\overline{A}, \overline{\Delta}, \overline{\varepsilon}, \overline{\Phi}, \overline{R})$ we understand an algebra homomorphism $f\colon A \to \overline{A}$ such that

$$\overline{\Delta} \circ f = (f \otimes f) \circ \Delta, \quad \overline{\varepsilon} \circ f = \varepsilon, \quad \overline{\Phi} = (f \otimes f \otimes f)(\Phi), \quad \overline{R} = (f \otimes f)(R)$$

(if we liked, we could have extended the class of morphisms to include twists). If $\mathfrak{g}$ is a deformation algebra, $F \in U\mathfrak{g} \hat{\otimes} U\mathfrak{g}$, $F \equiv 1 \bmod h$, and $(\varepsilon \otimes \mathrm{id})(F) = (\mathrm{id} \otimes \varepsilon)(F) = 1$, we denote by $A_{\mathfrak{g}, F}$ the triangular quasi-Hopf QUE-algebra obtained by twisting via F from the algebra $U\mathfrak{g}$ with the trivial triangular quasi-Hopf QUE-algebra structure. From Propositions 3.6 and 3.7 it follows that every object C is isomorphic to an object of the form $A_{\mathfrak{g}, F}$, so that what remains is to describe the morphisms $A_{\mathfrak{g}, F} \to A_{\overline{\mathfrak{g}}, \overline{F}}$.

Proposition 3.9. *Let λ be a homomorphism $\mathfrak{g} \to \overline{\mathfrak{g}}$, $u \in U\overline{\mathfrak{g}}$, $\varepsilon(u) = 1$, $u \equiv 1 \bmod h$, and $\overline{F} = (u \otimes u) \cdot (\tilde{\lambda} \otimes \tilde{\lambda})(F) \cdot \Delta(u)^{-1}$, where $\tilde{\lambda}$ is the extension of λ to a homomorphism $U\mathfrak{g} \to U\overline{\mathfrak{g}}$, while $\Delta\colon U\mathfrak{g} \to U\mathfrak{g} \hat{\otimes} U\mathfrak{g}$ is the standard (untwisted) comultiplication. Then the mapping $(\operatorname{Ad} u) \circ \tilde{\lambda}\colon U\mathfrak{g} \to U\overline{\mathfrak{g}}$ is a morphism $A_{\mathfrak{g}, F} \to A_{\overline{\mathfrak{g}}, \overline{F}}$, and all morphisms $A_{\mathfrak{g}, F} \to A_{\overline{\mathfrak{g}}, \overline{F}}$ are so obtained. Two pairs (λ, u) and (λ_1, u_1) determine the same morphism $A_{\mathfrak{g}, F} \to A_{\overline{\mathfrak{g}}, \overline{F}}$ if and only if $\lambda_1 = \exp(-h \cdot \operatorname{ad} v) \circ \lambda$ and $u_1 = ue^{hv}$ for some $v \in \overline{\mathfrak{g}}$.*

Proof. We show that to any morphism $f\colon A_{\mathfrak{g}, F} \to A_{\overline{\mathfrak{g}}, \overline{F}}$ corresponds some pair (λ, u) (the rest is obvious). Indeed, f is also a morphism $A_{\mathfrak{g}, 1} \to A_{\overline{\mathfrak{g}}, \tilde{F}}$, where $\tilde{F} = (f \otimes f)(F^{-1}) \cdot \overline{F}$. This means in particular that we have the equalities $\Phi = 1$ and $R = 1$ in the triangular quasi-Hopf QUE-algebra $A_{\overline{\mathfrak{g}}, \tilde{F}}$. It follows therefore from Proposition 3.8 that $\tilde{F} = (u \otimes u) \cdot \Delta(u)^{-1}$ for some $u \in U\overline{\mathfrak{g}}$ such that $\varepsilon(u) = 1$ and $u \equiv 1 \bmod h$. Furthermore, $\operatorname{Ad} u$ is an isomorphism $A_{\overline{\mathfrak{g}}, 1} \to A_{\overline{\mathfrak{g}}, \tilde{F}}$, so that $(\operatorname{Ad} u)^{-1} f$ is a morphism $A_{\mathfrak{g}, 1} \to A_{\overline{\mathfrak{g}}, 1}$; i.e., $(\operatorname{Ad} u)^{-1} f = \tilde{\lambda}$ for some homomorphism $\lambda\colon \mathfrak{g} \to \tilde{\mathfrak{g}}$. Since $(f \otimes f)(F^{-1}) \cdot \overline{F} = \tilde{F} = (u \otimes u) \cdot \Delta(u)^{-1}$ and $f = (\operatorname{Ad} u) \circ \tilde{\lambda}$ we have $\overline{F} = (u \otimes u) \cdot (\tilde{\lambda} \otimes \tilde{\lambda})(F)\Delta(u)^{-1}$. ●

Remark. $A_{\mathfrak{g}, F}$ is a Hopf QUE-algebra if and only if $F(x, y)F(x * y, z) = F(y, z)F(x, y * z)$. This equation (with F replaced by F^{-1}) is studied in [17].

We pass now to coboundary quasi-Hopf QUE-algebras. In contrast to the above triangular case, there arises here the *existence problem for the quantization of the pair* $(\mathfrak{g}, \varphi)$, where $\mathfrak{g}$ is a Lie algebra over k and $\varphi \in \bigwedge^3 \mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$

is a $\mathfrak{g}$-invariant tensor. Fortunately, the problem has a positive solution. To be specific, take $(U\mathfrak{g})[[h]]$ for A, define Δ and ε in the usual fashion, and put $R = 1$. It remains to construct a $\mathfrak{g}$-invariant element $\Phi \in (U\mathfrak{g} \otimes U\mathfrak{g} \otimes U\mathfrak{g})[[h]]$ satisfying (1.8), (1.10), (3.15) and the congruences $\Phi \equiv 1 \bmod h^2$, $\operatorname{Alt}\Phi \equiv h^2\varphi \bmod h^3$.

PROPOSITION 3.10. *Such a Φ exists.*

Before proving Proposition 3.10, let us observe that the complex (2.8) has an involution σ taking $a_1 \otimes \cdots \otimes a_n \in (U\mathfrak{g})^{\otimes n}$ into $(-1)^{n(n+1)/2} a_n \otimes \cdots \otimes a_1$. The subcomplex (2.8a) is mapped into itself by σ. Consider the complex

$$0 \to C^0 \to C^1 \to C^2 \to \cdots \tag{3.17}$$

where $C^n = \{x \in B^n | \sigma(x) = -x\}$, with B^n having the same meaning as in (2.8a).

PROPOSITION 3.11. 1) *The nth cohomology space of the complex* (3.17) *is equal to* 0 *for evern n, and isomorphic to $\bigwedge^n \mathfrak{g}$ for odd n (the isomorphism being induced by the natural imbedding $\bigwedge^n \mathfrak{g} \to C^n$).*

2) *If a $\mathfrak{g}$-invariant element of C^n is a coboundary, it is the coboundary of a $\mathfrak{g}$-invariant element of C^{n-1}.*

PROOF. On the cohomology H^n of the complex (2.8a) (i.e., on $\bigwedge^n \mathfrak{g}$), σ acts as multiplication by $(-1)^n$ (indeed, the sign of the permutation $(n, n-1, \dots, 1)$ is $(-1)^{n(n-1)/2} = (-1)^{n(n+1)/2}(-1)^n$). This implies part 1) of the proposition. To prove part 2), it suffices to show that the differential $d: C^{n-1} \to D^n$, where $D^n \subset C^n$ is the subspace of coboundaries, has a $\mathfrak{g}$-equivariant section $D^n \to C^{n-1}$. The complex (2.8a) is canonically isomorphic (see the proof of Proposition 2.2) to the direct sum of the complexes $\mathfrak{g}^{\otimes m} \otimes_{S_m} \overline{C}^*(I^m, \partial I^m)$, $m = 0, 1, 2, \dots$, where I^m is the simplicial m-cube and $\overline{C}^*$ denotes the normalized cochain complex with coefficients in $\mathbb{Q}$ (normalization means that the cochains are 0 on degenerate simplices). The involution σ comes from involutions τ_m of the complexes $\overline{C}^*(I^m, \partial I^m)$ (τ_m is induced by the automorphism of the "geometric" cube $[0, 1]^m$ that takes $(x_1, \dots, x_m)$ into $(1-x_1, \dots, 1-x_m)$). Denote by K_m the τ_m-anti-invariant part of the complex $\overline{C}^*(I^m, \partial I^m)$, by K_m^n the nth term of K_m, and by $D_m^n \subset K_m^n$ the subspace of coboundaries. The differential $d: K_m^{n-1} \to D_m^n$ has an S_m-equivariant section $s_m^n: D_m^n \to K_m^{n-1}$. The sections s_m^n, $m = n, n+1, \dots$, induce the desired section $D^n \to C^{n-1}$. ●

PROOF OF PROPOSITION 3.10. Suppose already constructed a $\mathfrak{g}$-invariant $\Phi_n \in (U\mathfrak{g} \otimes U\mathfrak{g} \otimes U\mathfrak{g})[[h]]$ such that $\Phi_n \equiv 1 \bmod h^2$, $\operatorname{Alt}\Phi_n \equiv h^2\varphi \bmod h^3$ and Φ_n satisfies modulo h^n the equations (1.8), (1.10), (3.15) (for Φ_3, or even Φ_4, one can take $1 + h^2\varphi/6$. Then $\Phi_n(x, y, 0) \equiv 1 \equiv \Phi_n(0, y, z) \bmod h^n$ (see the remark after formulas (1.7)–(1.10)). It is easy to construct a $\mathfrak{g}$-invariant Φ_n' such that $\Phi_n' \equiv \Phi_n \bmod h^n$, $\Phi_n'(0, y, z) \equiv \Phi_n'(x, 0, z) \equiv \Phi_n'(x, y, 0) \equiv 1 \bmod h^{n+1}$, and $\Phi_n'(z, y, x) \equiv \Phi_n'(x, y, z)^{-1} \bmod h^{n+1}$ (it suffices to put $\Phi_n' = e^L$, where L is obtained from $\ln \Phi_n \in (U\mathfrak{g} \otimes U\mathfrak{g} \otimes U\mathfrak{g})[[h]]$ by deleting the terms containing h^n). The element Φ_n' satisfies (1.8) modulo h^n, but not, in general, modulo h^{n+1}. Define a $\mathfrak{g}$-invariant element $\mu \in (U\mathfrak{g})^{\otimes 4}$ from the relation

$$\begin{aligned}\Phi_n'(x, y, z * u)\Phi_n'(x * y, z, u) &\equiv \Phi_n'(y, z, u)\Phi_n'(x, y * z, u) \\ &\quad \times \Phi_n'(x, y, z) + h^n \mu(x, y, z, u) \mod h^{n+1}.\end{aligned} \tag{3.18a}$$

We now try to find Φ_{n+1} in the form $\Phi_n' + h^n \psi$, $\psi \in (U\mathfrak{g})^{\otimes 3}$. The element ψ must be $\mathfrak{g}$-invariant and satisfy the equations

$$\psi(x, 0, z) = 0, \qquad \psi(z, y, x) = -\psi(x, y, z),$$
$$\begin{aligned}\psi(y, z, u) - \psi(x * y, z, u) + \psi(x, y * z, u) - \psi(x, y, z * u) + \psi(x, y, z) \\ = \mu(x, y, z, u).\end{aligned}$$

Proposition 3.11 shows that such a ψ exists, provided μ is a σ-anti-invariant cocycle of the complex (2.8a). We show that this is indeed the case. The equalities $\mu(0, y, z, u) = \mu(x, 0, z, u) = \mu(x, y, 0, u) = \mu(x, y, z, 0) = 0$ and $\mu(u, z, y, x) = -\mu(x, y, z, u)$ are easily verified. It remains to prove that μ is a cocycle, i.e., that

$$\begin{aligned}\mu(x, y, z, u) + \mu(x, y, z * u, v) + \mu(x * y, z, u, v) \\ = \mu(y, z, u, v) + \mu(x, y * z, u, v) + \mu(x, y, z, u * v).\end{aligned} \tag{3.19}$$

For this we use, along with (3.18a), the congruences (3.18b)–(3.18f) obtained from (3.18a) by replacing x, y, z, u by $x, y, z * u, v$ in case b), $x * y, z, u, v$ in case c), y, z, u, v in case d), $x, y*z, u, v$ in case e), and $x, y, z, u*v$ in case f). We now compute the products

$$\begin{aligned}&\{\Phi_n'(x, y * z, u)\Phi_n'(x, y, z)\} \\ &\times \{\Phi_n'(x * y * z, u, v)^{-1}\Phi_n'(x * y, z, u * v)^{-1}\Phi_n'(z, u, v)\} \\ &\times \{\Phi_n'(x, y, z * u * v)^{-1}\Phi_n'(y, z * u, v)\Phi_n'(x, y * z * u, v)\} \\ &\times \Phi_n'(y, z, u),\end{aligned} \tag{3.20}$$

$$\begin{aligned}&\{\Phi_n'(x, y * z, u)\Phi_n'(x * y * z, u, v)^{-1}\} \\ &\times \{\Phi_n'(x, y, z)\Phi_n'(x * y, z, u * v)^{-1}\Phi_n'(x, y, z * u * v)^{-1}\} \\ &\times \{\Phi_n'(z, u, v)\Phi_n'(y, z * u, v)\Phi_n'(y, z, u)\} \\ &\times \Phi_n'(x, y * z * u, v).\end{aligned} \tag{3.21}$$

Rewriting the curly brackets in (3.20) by using (3.18a), (3.18c), (3.18b), and the curly brackets in (3.21) by using (3.18e), (3.18f), (3.18d), we find that

$$\begin{aligned}(3.20) \equiv 1 - h^n(\mu(x, y, z, u) + \mu(x, y, z * u, v) \\ + \mu(x * y, z, u, v)) \mod h^{n+1}\end{aligned}$$

and

$$\begin{aligned}(3.21) \equiv 1 - h^n(\mu(y, z, u, v) + \mu(x, y * z, u, v) \\ + \mu(x, y, z, u * v)) \mod h^{n+1}.\end{aligned}$$

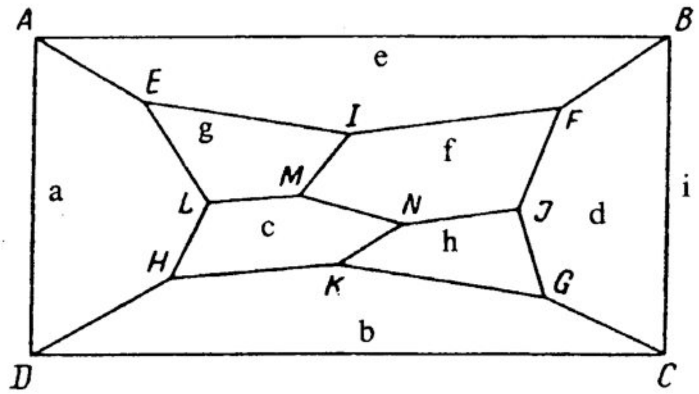
On the other hand, the equalities

$$\Phi'_n(x, y, z)\Phi'_n(x*y*z, u, v) = \Phi'_n(x*y*z, u, v)\Phi'_n(x, y, z), \tag{3.18g}$$

$$\Phi'_n(z, u, v)\Phi'_n(x, y, z*u*v) = \Phi'_n(x, y, z*u*v)\Phi'_n(z, u, v), \tag{3.18h}$$

$$\Phi'_n(y, z, u)\Phi'_n(x, y*z*u, v) = \Phi'_n(x, y*z*u, v)\Phi'_n(y, z, u), \tag{3.18i}$$

which follow from the $\mathfrak{g}$-invariance of Φ'_n, show that (3.20) = (3.21). This implies (3.19).



Let us observe that the derivation of (3.19) can be pictured graphically by considering the boundary L_5 of the complex K_5 constructed by Stasheff in [18]. L_5 is the sphere S^2 divided into rectangular and pentagonal faces. The stereographic projection of this subdivision is indicated in the figure (the face "i" contains the point at infinity). The vertices $A, B, \ldots, N$ correspond to the 14 ways of arranging parentheses in a product of five factors x, y, z, u, v; namely,

$$\begin{array}{lll} A = (x((yz)u))v, & B = x(((yz)u)v), & C = x((y(zu))v), \\ D = (x(y(zu)))v, & E = ((x(yz))u)v, & F = x((yz)(uv)), \\ G = x(y(zu)v)), & H = ((xy)(zu))v, & I = (x(yz))(uv), \\ J = x(y(z(uv))), & K = (xy)((zu)v), & L = (((xy)z)u)v, \\ M = ((xy)z)(uv), & N = (xy)(z(uv)). & \end{array}$$

Two vertices are joined by an edge if the corresponding parenthesis arrangements are obtained one from the other by a single application of the associative law. To each oriented edge we assign an element of the group G_1/G_{n+1}, where G_k is the set $\{a \in (U\mathfrak{g} \otimes U\mathfrak{g} \otimes U\mathfrak{g})[[h]] | a \equiv 1 \bmod h^k\}$ under the operation of multiplication. The rule for this assignment is indicated by the following example: the edge AE corresponds to the associativity relation $x((yz)u) = (x(yz))u$, where A corresponds to the "right" arrangement $x((yz)u)$ and E to the "left" arrangement $(x(yz))u$; accordingly, we assign the element $\Phi'_n(x, y*z, u)$ to the edge AE, and the element $\Phi'_n(x, y*z, u)^{-1}$ to the edge EA. Relations (3.18a)–(3.18i) show that for each of the faces "a", "b", ... ,"i" the product of the elements of G_1/G_{n+1} corresponding to the edges of the face taken all in the same direction belongs to the subgroup G_n/G_{n+1}, which is contained in the center of G_1/G_{n+1}. We call this product a residual. By

(3.18g)–(3.18i), the residuals corresponding to rectangular faces are equal to 1. Therefore the product of the residuals corresponding to the pentagonal faces (running, say, clockwise over all of them) is also equal to 1, and this is equivalent to (3.19). ●

Let us take fixed sections s_m^4, $m = 4, 5, \ldots$ (see the end of the proof of Proposition 3.11). Then the proof of Proposition 3.10 presents a completely specific Φ, expressed in terms of $h^2\varphi$ by means of a "universal formula", which for short we shall write in the form $\Phi = \mathscr{E}(h^2\varphi)$. The words "universal formula" mean that if we write Φ in the form

$$\sum_{m,n,p=0}^{\infty} a_{(m,n,p)}^{i_1,\ldots,i_m j_1,\ldots,j_n k_1,\ldots,k_p} e_{i_1},\ldots,e_{i_m}\otimes e_{j_1},\ldots,e_{j_n}\otimes e_{k_1},\ldots,e_{k_p},$$

where the e_i are a basis in $\mathfrak{g}$ and the tensors $a_{(m,n,p)}$ are symmetric in each of the groups of indices i, j, k, then the $a_{(m,n,p)}$ can be expressed in terms of the structural constants C_{rs}^t of the algebra $\mathfrak{g}$ and the components ψ^{uvw} of the tensor $\psi = h^2\varphi$ in accordance with the rules of acyclic tensor calculus with coefficients in $\mathbb{Q}$, while the relations between the tensors $a_{(m,n,p)}$, equivalent to (1.8), (1.10), and (3.15), follow in accordance with the rules of acyclic tensor calculus from the relations $c_{ij}^k = -c_{ji}^k$, $c_{ij}^r c_{kr}^l + c_{jk}^r c_{ir}^l + c_{ki}^r c_{jr}^l = 0$, $\psi^{ijk} = -\psi^{jik} = -\psi^{ikj}$, $c_{rs}^i\psi^{sjk} + c_{rs}^j\psi^{isk} + c_{rs}^k\psi^{ijs} = 0$. Acyclicity (meaning, for example, exclusion of the expression $c_{ri}^j c_{sj}^k c_{tk}^i$, where i, j, k form a "cycle") ensures meaningfulness for the formula $\mathscr{E}$ in the infinite-dimensional case and in the case that k is not a field and the k-module $\mathfrak{g}$ is nonfree. We note that $\mathscr{E}(\psi)$ is of the form $1 + \psi/6 + o(\psi)$, where $o(\psi)$ denotes terms in powers of ψ greater than 1.

PROPOSITION 3.12. *Suppose given a $\mathfrak{g}$-invariant $\Phi \in (U\mathfrak{g}\otimes U\mathfrak{g}\otimes U\mathfrak{g})[[h]]$ satisfying* (1.8), (1.10), (3.15) *and the congruences* $\Phi \equiv 1 \bmod h$, $\operatorname{Alt}\Phi \equiv h^2\varphi \bmod h^3$, $\varphi \in \bigwedge^3\mathfrak{g}$. *Then there exists a $\mathfrak{g}$-invariant symmetric $F \in (U\mathfrak{g}\otimes U\mathfrak{g})[[h]]$ such that $F \equiv 1 \bmod h$, $(\varepsilon\otimes\mathrm{id})(F) = 1 = (\mathrm{id}\otimes\varepsilon)(F)$, and $\tilde{\Phi}$, as defined by formula* (1.12), *is equal to $\mathscr{E}(\psi)$, where ψ is a $\mathfrak{g}$-invariant element of $(\bigwedge^3\mathfrak{g})[[h]]$ congruent to $h^2\varphi \bmod h^3$. Here ψ is defined uniquely, and F is defined up to multiplication by an element of the form $(u^{-1}\otimes u^{-1})\Delta(u)$, where u belongs to the center of $(U\mathfrak{g})[[h]]$, $u \equiv 1 \bmod h$, and $\varepsilon(u) = 1$.*

PROOF. It suffices to use Proposition 3.11 (for $n = 3$ to prove existence, and for $n = 2$ to prove uniqueness). ●

REMARK. Let $\overline{\mathscr{E}}(\psi)$ be an arbitrary "universal" solution of equations (1.8), (1.10), (3.15) of the form $\overline{\mathscr{E}}(\psi) = 1 + \psi/6 + o(\psi)$ (there *are* such $\overline{\mathscr{E}}$ that are no less "natural" then $\mathscr{E}$; for example, the particular sections s_m^4 in the definition of $\mathscr{E}$ can be replaced by others, or Φ_n' can be constructed from Φ_n in some other way than indicated in the proof of Proposition 3.10). Then, arguing as in the proof of Proposition 3.12, we find that $\overline{\mathscr{E}}(\psi)$ can be reduced, by twisting via a symmetric "universal" $F(\psi)$ of the form $1 + o(\psi)$, to the form $\mathscr{E}(\tilde{\psi})$, where $\tilde{\psi}$ is expressed in terms of ψ by means of a "universal formula" of the form $\tilde{\psi} = \psi + o(\psi)$. Here $\tilde{\psi}$ is defined uniquely for the given $\overline{\mathscr{E}}$, and $F(\psi)$ up to multiplication by $(u^{-1}\otimes u^{-1})\Delta(u)$, where u is expressed in terms of ψ by means of a "universal formula" of the form $u = 1 + o(\psi)$. An example of a nontrivial "renormalization" of ψ is given by $\tilde{\psi}^{ijk} = \psi^{ijk} + a\,\mathrm{Alt}_{i,j,k}\psi^{ilm}\psi^{jpq}c^n_{lm}c^r_{pq}c^k_{nr}$, where c is the tensor of structural constants of $\mathfrak{g}$, and $a \in \mathbb{Q}$.

Now let $\mathfrak{g}$ be a deformation algebra over $k[[h]]$, and φ a $\mathfrak{g}$-invariant element of $\bigwedge^3 \mathfrak{g}$, where $\bigwedge^3 \mathfrak{g}$ denotes the skewsymmetric part of $\mathfrak{g}\hat{\otimes}\mathfrak{g}\hat{\otimes}\mathfrak{g}$. We can make $U\mathfrak{g}$ into a coboundary quasi-Hopf QUE-algebra by defining the usual comultiplication in it and putting $R = 1$, $\Phi = \mathscr{E}(h^2\varphi)$. Then twisting via an element $F \in U\mathfrak{g}\hat{\otimes}U\mathfrak{g}$ such that $F \equiv 1 \bmod h$ and $(\varepsilon\otimes\mathrm{id})(F) = (\mathrm{id}\otimes\varepsilon)(F) = 1$, we obtain again a coboundary quasi-Hopf QUE-algebra, which we denote by $A_{\mathfrak{g},\varphi,F}$.

PROPOSITION 3.13. *Any coboundary quasi-Hopf* QUE-*algebra is isomorphic to* $A_{\mathfrak{g},\varphi,F}$ *for some* $\mathfrak{g}$, φ, F.

PROOF. Everything reduces to the following lemma.

LEMMA. *Let $\bar{\mathfrak{a}}$ be a Lie algebra over $k[h]/(h^n)$, $n \geq 1$, and a free module over $k[h]/(h^n)$, and let $\overline{\psi} \in \bigwedge^3\bar{\mathfrak{a}}$ be invariant with respect to the adjoint representation, with $\overline{\psi} \equiv 0 \bmod h$. Let $(A, \Delta, \varepsilon, \Phi, R)$ be a coboundary quasi-Hopf algebra over $k[h]/(h^{n+1})$ and a free module over $k[h]/(h^{n+1})$, such that after reduction $\bmod h^n$ the set (A, Δ, ε) becomes $U\bar{\mathfrak{a}}$ with the usual Hopf structure, while Φ and R become $\mathscr{E}(\psi)$ and 1. Then by twisting $(A, \Delta, \varepsilon, \Phi, R)$ via some $F \in A\otimes A$ such that $F \equiv 1 \bmod h^n$ and $(\varepsilon\otimes\mathrm{id})(F) = (\mathrm{id}\otimes\varepsilon)(F) = 1$, we can make $(A, \Delta, \varepsilon) = U\mathfrak{a}$, $\Phi = \mathscr{E}(\psi)$, $R = 1$, where $\mathfrak{a}$ is a Lie algebra over $k[h]/(h^{n+1})$, a free module over $k[h]/(h^{n+1})$ and such that $\mathfrak{a}/h^n\mathfrak{a} = \bar{\mathfrak{a}}$, while $\psi \in \bigwedge^3\mathfrak{a}$ is invariant with respect to the adjoint representation of $\mathfrak{a}$, with $\psi \bmod h^n = \overline{\psi}$.*

PROOF. Put $L = k[h]/(h^n)$, $I_r = h^r k[h]/h^{n+r-1}k[h]$, $\tilde{I}_1 = hk[h]/h^{n+1}k[h]$. Choosing for $\overline{\psi} \in (\bigwedge^3\bar{\mathfrak{a}})\otimes_L I_1$ an inverse image $\tilde{\psi} \in (\bigwedge^3\bar{\mathfrak{a}})\otimes_L \tilde{I}_1$ (which is not, in general, $\bar{\mathfrak{a}}$-invariant), we show that we can properly define $\Phi_1 = \mathscr{E}(\tilde{\psi}) \in A\otimes A\otimes A$, satisfying (1.8), (1.10), (3.15). For this, consider the L-algebras $B = L\oplus I_1 t\oplus I_2 t^2\oplus\cdots$, $\tilde{B} = L\oplus\tilde{I}_1 t\oplus I_2 t^2 + I_3 t^3\oplus\cdots$ (where t is a formal variable) and the element $\overline{\psi}t \in (\bigwedge^3\bar{\mathfrak{a}})\otimes_L B$. We have $\mathscr{E}(\overline{\psi}t) \in (U\bar{\mathfrak{a}}\otimes U\bar{\mathfrak{a}}\otimes U\bar{\mathfrak{a}})\otimes_L B$ and $\mathscr{E}(\overline{\psi}t) - \overline{\psi}t - 1 \in (U\bar{\mathfrak{a}}\otimes U\bar{\mathfrak{a}}\otimes U\bar{\mathfrak{a}})\otimes_L J$, where $J = I_2t^2\oplus I_3t^3\oplus\cdots$. This allows us to define $\mathscr{E}(\tilde{\psi}t) \in (U\bar{\mathfrak{a}}\otimes U\bar{\mathfrak{a}}\otimes U\bar{\mathfrak{a}})\otimes_L\tilde{B}$ by the formula $\mathscr{E}(\tilde{\psi}t) = (\mathscr{E}(\overline{\psi}t) - \overline{\psi}t) + \tilde{\psi}t$ and verify that $\mathscr{E}(\tilde{\psi}t)$ satisfies (1.8), (1.10), (3.15). Finally, put $\Phi_1 = 1 + (f_1\circ f_2)(\mathscr{E}(\tilde{\psi}t) - 1)$, where f_1 is the natural mapping $(U\bar{\mathfrak{a}}\otimes U\bar{\mathfrak{a}}\otimes U\bar{\mathfrak{a}})\otimes_L\tilde{I}_1 \to A\otimes A\otimes A$, while f_2: $(U\bar{\mathfrak{a}}\otimes U\bar{\mathfrak{a}}\otimes U\bar{\mathfrak{a}})\otimes_L(\tilde{I}_1t\oplus I_2t^2\oplus I_3t^3\oplus\cdots) \to (U\bar{\mathfrak{a}}\otimes U\bar{\mathfrak{a}}\otimes U\bar{\mathfrak{a}})\otimes_L\tilde{I}_1$ is induced by the natural mapping $\tilde{I}_1t\oplus I_2t^2\oplus I_3t^3\oplus\cdots\to\tilde{I}_1$ (putting t equal to 1). Clearly, $\Phi_1 \equiv \Phi \bmod h^n$.

We proceed now to the twist of $(A, \Delta, \varepsilon, \Phi, R)$. First we make $R = 1$. Then Φ satisfies, along with (1.8) and (1.10), the relation (3.15), so that $h^{-n}(\Phi - \Phi_1) \bmod h$ is a 3-cocycle of the complex (3.17). A further twisting of $(A, \Delta, \varepsilon, \Phi, R)$ by means of symmetric elements $F \in A\otimes A$ such that $F \equiv 1 \bmod h^n$ and $(\varepsilon\otimes\mathrm{id})(F) = (\mathrm{id}\otimes\varepsilon)(F) = 1$ allows us to change this cocycle by a coboundary without destroying the condition $R = 1$. We can therefore (see Proposition 3.11) arive at the relation

$$\Phi = \Phi_1 + h^n\chi, \qquad \chi \in \bigwedge^3(\bar{\mathfrak{a}}/h\bar{\mathfrak{a}}). \tag{3.22}$$

Since $R = 1$, Δ is cocommutative. Hence

$$\mathrm{Alt}\{(\Delta\otimes\mathrm{id})(\Delta(a)) - (\mathrm{id}\otimes\Delta)(\Delta(a))\} = 0, \qquad a \in A. \tag{3.23}$$

On the other hand, from (1.1), (3.22), and the definition of Φ_1 it follows that if the image of a in $A/h^nA = U\bar{\mathfrak{a}}$ is $\alpha \in \bar{\mathfrak{a}}$, then

$$\begin{aligned}(\Delta\otimes\mathrm{id})&(\Delta(a)) - (\mathrm{id}\otimes\Delta)(\Delta(a))\\ &= [\alpha\otimes 1\otimes 1 + 1\otimes\alpha\otimes 1 + 1\otimes 1\otimes\alpha,\ \tilde{\psi} + h^n\chi].\end{aligned} \tag{3.24}$$

From (3.23) and (3.24) it follows that both sides of (3.24) are equal to 0. Therefore Δ is coassociative. Arguing as in the proof of Proposition 3.7, we find that $(A, \Delta, \varepsilon) = U\mathfrak{a}$ for some Lie algebra $\mathfrak{a}$ over $k[h]/(h^{n+1})$ such that $\mathfrak{a}$ is a free module over $k[h]/(h^{n+1})$ and $\mathfrak{a}/h^n\mathfrak{a} = \bar{\mathfrak{a}}$. Identify $(\bigwedge^3\bar{\mathfrak{a}})\otimes_L I_1$ with $h\cdot\bigwedge^3\mathfrak{a}$ and put $\psi = \tilde{\psi} + h^n\chi \in h\cdot\bigwedge^3\mathfrak{a}$. Since the right-hand side of (3.24) is equal to 0, the element ψ is $\mathfrak{a}$-invariant. It is clear that $\mathscr{E}(\psi) = \Phi$. ●

We describe now the morphisms $A_{\mathfrak{g},\varphi,F} \to A_{\bar{\mathfrak{g}},\bar{\varphi},\bar{F}}$, in the spirit of Proposition 3.9.

Proposition 3.14. *Let λ be a homomorphism $\mathfrak{g} \to \bar{\mathfrak{g}}$ such that $(\lambda\otimes\lambda\otimes\lambda)(\varphi) = \bar{\varphi}$; let $u \in U\bar{\mathfrak{g}}$ be such that $\varepsilon(u) = 1$, $u \equiv 1 \bmod h$, and $\bar{F} = (u\otimes u)\cdot(\tilde{\lambda}\otimes\tilde{\lambda})(F)\cdot\Delta(u)^{-1}$ where $\tilde{\lambda}$ is the extension of λ to a homomorphism $U\mathfrak{g} \to U\bar{\mathfrak{g}}$, and $\Delta\colon U\mathfrak{g} \to U\mathfrak{g}\hat{\otimes}U\mathfrak{g}$ is the standard (untwisted) comultiplication. Then the mapping $(\mathrm{Ad}\,u)\circ\tilde{\lambda}\colon U\mathfrak{g} \to U\bar{\mathfrak{g}}$ is a morphism $A_{\mathfrak{g},\varphi,F} \to A_{\bar{\mathfrak{g}},\bar{\varphi},\bar{F}}$, and all morphisms $A_{\mathfrak{g},\varphi,F} \to A_{\bar{\mathfrak{g}},\bar{\varphi},\bar{F}}$ are so obtained. Two pairs (λ, u) and (λ_1, u_1) define the same morphism $A_{\mathfrak{g},\varphi,F} \to A_{\bar{\mathfrak{g}},\bar{\varphi},\bar{F}}$ if and only if $\lambda_1 = \exp(-h\cdot\mathrm{ad}\,v)\circ\lambda$ and $u_1 = ue^{hv}$ for some $v \in \bar{\mathfrak{g}}$.*

Proof. As in the proof of Proposition 3.9, it suffices to verify that if there exists a homomorphism $A_{\mathfrak{g},\varphi,1} \to A_{\bar{\mathfrak{g}},\bar{\varphi},\bar{F}}$ then $\bar{F}$ has the form $(u\otimes u)\cdot\Delta(u^{-1})$, where $u \in U\bar{\mathfrak{g}}$, $\varepsilon(u) = 1$, and $u \equiv 1 \bmod h$. For this in turn it suffices to construct elements $u_n \in U\bar{\mathfrak{g}}$, $n \geq 1$, such that $\varepsilon(u_n) = 1$, $u_n \equiv 1 \bmod h$, $\bar{F} \equiv (u_n\otimes u_n)\cdot\Delta(u_n^{-1}) \bmod h^n$, and $u_{n+1} \equiv u_n \bmod h^n$. For u_1 we can take 1. Suppose u_n already constructed, and put $\tilde{F} = (u_n^{-1}\otimes u_n^{-1})\cdot\bar{F}\cdot\Delta(u_n)$. Then $\tilde{F} \equiv 1 \bmod h^n$, while $\mathrm{Ad}\,u_n$ is an isomorphism $A_{\bar{\mathfrak{g}},\bar{\varphi},\bar{F}} \to A_{\bar{\mathfrak{g}},\bar{\varphi},\tilde{F}}$. Hence there exists a homomorphism $f\colon A_{\mathfrak{g},\varphi,1} \to A_{\bar{\mathfrak{g}},\bar{\varphi},\tilde{F}}$. It follows that in the coboundary quasi-Hopf QUE-algebra $A_{\bar{\mathfrak{g}},\bar{\varphi},\tilde{F}}$ the element R is equal to 1, so that $\tilde{F}$ is symmetric. Put $\theta = h^{-n}(\tilde{F} - 1) \bmod h$; then θ belongs to the term C^2 of the complex (3.17). The proof that u_{n+1} exists now reduces to verifying that θ is a coboundary. By Proposition 3.11, it suffices to prove that $d\theta = 0$, where d is the differential of the complex (3.17).

The reduction of f modulo h^n is a Hopf algebra homomorphism $U\mathfrak{g}/h^nU\mathfrak{g} \to U\bar{\mathfrak{g}}/h^nU\bar{\mathfrak{g}}$ and therefore maps $\mathfrak{g}/h^n\mathfrak{g}$ into $\bar{\mathfrak{g}}/h^n\bar{\mathfrak{g}}$. Denote the image of φ in $\bigwedge^3(\bar{\mathfrak{g}}/h^n\bar{\mathfrak{g}})$ by $\boldsymbol{\varphi}$. Then $(f\otimes f\otimes f)(\mathscr{E}(h^2\varphi)) \bmod h^{n+2} = \mathscr{E}(h^2\boldsymbol{\varphi})$. On the other hand, $(f\otimes f\otimes f)(\mathscr{E}(h^2\varphi))$ is obtained from $\mathscr{E}(h^2\bar{\varphi})$ by twisting via $\tilde{F}$.

Hence $\mathscr{E}(h^2\varphi)-\mathscr{E}(h^2\overline{\varphi}) \equiv h^n \cdot d\theta \bmod h^{n+1}$. It follows that $h^2(\varphi-\overline{\varphi}) \equiv h^n\, d\theta \bmod h^{n+1}$, so that $d\theta \in \bigwedge^3(\mathfrak{g}/h\mathfrak{g})$. This can happen only when $d\theta = 0$. •

Finally, we come to the quasitriangular quasi-Hopf QUE-algebras. It turns out that any pair $(\mathfrak{g}, t)$, where $\mathfrak{g}$ is a Lie algebra over k and t a $\mathfrak{g}$-invariant element of $\mathrm{Sym}^2\mathfrak{g}$, can be quantized. To do this, we take $(U\mathfrak{g})[[h]]$ for A, define Δ and ε in the usual way, put $R = e^{ht/2}$, and try to find a $\mathfrak{g}$-invariant element $\Phi \in (U\mathfrak{g}\otimes U\mathfrak{g}\otimes U\mathfrak{g})[[h]]$ satisfying (1.8), (1.10), (3.9a), (3.15) (it will then also satisfy (3.9b), since $R(x, y*z) = R(z*y, x)$).

THEOREM 3.15. *Such a Φ exists and is unique up to twisting via symmetric $\mathfrak{g}$-invariant elements $F \in (U\mathfrak{g}\otimes U\mathfrak{g})[[h]]$. It can be expressed in terms of $\tau = ht$ by means of a "universal formula" $\Phi = \mathscr{M}(\tau) = 1 + o(\tau)$, which is unique up to twisting via a symmetric "universal" $F(\tau)$ of the form $1 + o(\tau)$. Analogues of Propositions* 3.13 *and* 3.14 *hold also in this quasitriangular case.*

To keep the size of the present paper within bounds, the author is forced to postpone the proof of this theorem to a later publication. Here, we show that when $k = \mathbb{C}$ the element Φ can be determined from the relation $G_1 = G_2\Phi$, where G_1 and G_2 are solutions of the differential equation

$$G'(x) = \overline{h}\left(\frac{t^{12}}{x} + \frac{t^{23}}{x-1}\right) G(x), \qquad \overline{h} = h/2\pi i, \tag{3.25}$$

defined for $0 < x < 1$ with the asymptotic behavior $G_1(x) \sim x^{\overline{h}t^{12}}$ as $x \to 0$ and $G_2(x) \sim (1-x)^{\overline{h}t^{23}}$ as $x \to 1$. Here $t^{12} = t\otimes 1 \in (U\mathfrak{g})^{\otimes 3}$, $t^{23} = 1\otimes t \in (U\mathfrak{g})^{\otimes 3}$, $G(x) \in (U\mathfrak{g})^{\otimes 3}[[h]]$, $x^{\overline{h}t^{12}}$ should be interpreted as $\exp(\overline{h}\ln x \cdot t^{12}) = 1 + \overline{h}\ln x \cdot t^{12} + \cdots$, and the notation $G_1(x) \sim x^{\overline{h}t^{12}}$ means that $G_1(x) = (1 + f_1(x)h + f_2(x)h^2 + \cdots)x^{\overline{h}t^{12}}$, where the f_i are analytic at $x = 0$ and $f_i(0) = 0$. To prove (1.2) and (3.9a), consider the system

$$\frac{\partial W}{\partial z_i} = \overline{h}\sum_{j\neq i}\frac{t^{ij}}{z_i - z_j}\cdot W, \qquad i = 1, 2, \ldots, n, \tag{3.26}$$

where $W(z_1, \ldots, z_n) \in (U\mathfrak{g})^{\otimes n}[[h]]$ and t^{ij} is the image of t under the (i, j)th imbedding $U\mathfrak{g}\otimes U\mathfrak{g} \to (U\mathfrak{g})^{\otimes n}$. This system arose [19] in connection with the conformal field theory corresponding to current algebra, and has been studied in [5]–[8]. For us it is essential that, as indicated in [19], the system (3.26) is self-consistent, i.e., the curvature of the corresponding connection is equal to 0. We note also that $\partial W/\partial z_1 + \cdots + \partial W/\partial z_n = 0$, so that W depends only on the differences $z_i - z_j$.

For $n = 3$, the solutions of the system (3.26) are of the form

$$(z_1 - z_3)^{\overline{h}(t^{12}+t^{13}+t^{23})} G((z_1 - z_2)/(z_1 - z_3)),$$

where G satisfies (3.25). Hence Φ can be determined from the relation $W_1 = W_2\Phi$, where W_1 and W_2 are solutions of (3.26) for $n = 3$ in the domain $\{(z_1, z_2, z_3) \in \mathbb{R}^3 | z_1 > z_2 > z_3\}$ with $W_1 \sim (z_1 - z_2)^{\overline{h}t^{12}}(z_1 - z_3)^{\overline{h}(t^{13}+t^{23})}$ for

$z_1 - z_2 \ll z_1 - z_3$, and $W_2 \sim (z_2 - z_3)^{\bar{h}t^{23}}(z_1 - z_3)^{\bar{h}(t^{12}+t^{13})}$ for $z_2 - z_3 \ll z_1 - z_3$. To prove (1.2), consider the system (3.26) for $n = 4$ in the domain $\{(z_1, z_2, z_3, z_4) \in \mathbb{R}^4 | z_1 > z_2 > z_3 > z_4\}$, in which we distinguish five zones:

1) $z_1 - z_2 \ll z_1 - z_3 \ll z_1 - z_4$,
2) $z_2 - z_3 \ll z_1 - z_3 \ll z_1 - z_4$,
3) $z_2 - z_3 \ll z_2 - z_4 \ll z_1 - z_4$,
4) $z_3 - z_4 \ll z_2 - z_4 \ll z_1 - z_4$,
5) $z_1 - z_2 \ll z_1 - z_4$, $z_3 - z_4 \ll z_1 - z_4$.

These correspond to possible arrangements of parentheses in a product of symbols x_1, x_2, x_3, x_4:

1) $((x_1x_2)x_3)x_4$,
2) $(x_1(x_2x_3))x_4$,
3) $x_1((x_2x_3)x_4)$,
4) $x_1(x_2(x_3x_4))$,
5) $(x_1x_2)(x_3x_4)$.

It is easily shown that there exist solutions $W_1, \dots, W_5$ having the following asymptotic behaviors in the corresponding zones:

$$\begin{aligned}
W_1 &\sim (z_1 - z_2)^{\bar{h}t^{12}}(z_1 - z_3)^{\bar{h}(t^{13}+t^{23})}(z_1 - z_4)^{\bar{h}(t^{14}+t^{24}+t^{34})},\\
W_2 &\sim (z_2 - z_3)^{\bar{h}t^{23}}(z_1 - z_3)^{\bar{h}(t^{12}+t^{13})}(z_1 - z_4)^{\bar{h}(t^{14}+t^{24}+t^{34})},\\
W_3 &\sim (z_2 - z_3)^{\bar{h}t^{23}}(z_2 - z_4)^{\bar{h}(t^{24}+t^{34})}(z_1 - z_4)^{\bar{h}(t^{12}+t^{13}+t^{14})},\\
W_4 &\sim (z_3 - z_4)^{\bar{h}t^{34}}(z_2 - z_4)^{\bar{h}(t^{23}+t^{24})}(z_1 - z_4)^{\bar{h}(t^{12}+t^{13}+t^{14})},\\
W_5 &\sim (z_1 - z_2)^{\bar{h}t^{12}}(z_3 - z_4)^{\bar{h}t^{34}}(z_1 - z_4)^{\bar{h}(t^{13}+t^{14}+t^{23}+t^{24})}.
\end{aligned}$$

This should be understood, say for W_5, as meaning the equality $W_5 = (1 + f_1(u, v)h + f_2(u, v)h^2 + \cdots)(z_1 - z_2)^{\bar{h}t^{12}}(z_3 - z_4)^{\bar{h}t^{34}}(z_1 - z_4)^{\bar{h}(t^{13}+t^{14}+t^{23}+t^{24})}$, where $u = (z_1 - z_2)/(z_1 - z_4)$, $v = (z_3 - z_4)/(z_1 - z_4)$, and the f_i are analytic in a neighborhood of $(0, 0)$, with $f_i(0, 0) = 0$. It is easily shown that

$$\begin{aligned}
&W_1 = W_2 \cdot (\Phi \otimes 1), && W_2 = W_3 \cdot (\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Phi),\\
&W_3 = W_4 \cdot (1 \otimes \Phi), && W_1 = W_5 \cdot (\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\Phi),\\
&W_5 = W_4 \cdot (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi).
\end{aligned}$$

From all this follows (1.2).

To prove (3.9a), consider (3.26) for $n = 3$ in the domain $\{(z_1, z_2, z_3) \in \mathbb{C}^3 | \operatorname{Im} z_1 \geq \operatorname{Im} z_2 \geq \operatorname{Im} z_3, z_1 \neq z_2 \neq z_3 \neq z_1\}$. Let $W_1, \dots, W_6$ be solutions for which

$$\begin{aligned}
W_1 &\sim (z_1 - z_2)^{\bar{h}t^{12}}(z_1 - z_3)^{\bar{h}(t^{13}+t^{23})} && \text{for } |z_1 - z_2| \ll |z_1 - z_3|,\\
W_2 &\sim (z_2 - z_3)^{\bar{h}t^{23}}(z_1 - z_3)^{\bar{h}(t^{12}+t^{13})} && \text{for } |z_2 - z_3| \ll |z_1 - z_3|,\\
W_3 &\sim (z_3 - z_2)^{\bar{h}t^{23}}(z_1 - z_2)^{\bar{h}(t^{12}+t^{13})} && \text{for } |z_2 - z_3| \ll |z_1 - z_2|,\\
W_4 &\sim (z_1 - z_3)^{\bar{h}t^{13}}(z_1 - z_2)^{\bar{h}(t^{12}+t^{23})} && \text{for } |z_3 - z_1| \ll |z_1 - z_2|,\\
W_5 &\sim (z_3 - z_1)^{\bar{h}t^{13}}(z_3 - z_2)^{\bar{h}(t^{12}+t^{23})} && \text{for } |z_3 - z_1| \ll |z_3 - z_2|,\\
W_6 &\sim (z_1 - z_2)^{\bar{h}t^{12}}(z_3 - z_2)^{\bar{h}(t^{13}+t^{23})} && \text{for } |z_1 - z_2| \ll |z_1 - z_3|.
\end{aligned}$$

Suppose first that $z_i \in \mathbb{R}$. Considering W_1 and W_2 for $z_1 > z_2 > z_3$, W_3 and W_4 for $z_1 > z_3 > z_2$, W_5 and W_6 for $z_3 > z_1 > z_2$, we find that $W_1 = W_2\Phi$, $W_4 = W_3 \cdot \Phi^{132}$, $W_5 = W_6 \cdot \Phi^{312}$. On the other hand, considering complex values of the z_i, we find that $W_2 = W_3 e^{ht^{23}/2}$, $W_4 = W_5 e^{ht^{13}/2}$, $W_1 = W_6 e^{h(t^{13}+t^{23})/2}$. From this follows (3.9a).

REMARK ON KNOT INVARIANTS. Let $\mathfrak{g}$ and t be as above. By Theorem 3.15, the pair $(\mathfrak{g}, t)$ determines a quasitriangular quasi-Hopf algebra (A, Δ, Φ, R), where $A = (U\mathfrak{g})[[h]]$, Δ is the usual comultiplication, $R = e^{ht/2}$, and Φ is expressed in terms of $\tau = ht$ by the "universal formula" $\Phi = \mathscr{M}(\tau)$. Applying now the quasi-Hopf modification of the Reshetikhin construction (see the Introduction), we find that to every knot γ in $\mathbb{R}^3$ corresponds an element $p_\gamma \in (U\mathfrak{g})[[h]]$, expressed by a "universal formula" $p_\gamma = \mathscr{P}_\gamma(\tau)$. The "universal knot invariant" $\mathscr{P}_\gamma$ is independent of the arbitrariness in the choice of $\mathscr{M}$ and contains all invariants of R-matrix type that correspond to R-matrices belonging to the irreducible components of the solution manifold of equation (3.4) containing the identity matrix. Let V_r be the space of symmetric tensors of rank r formed from τ and the tensor of structural constants of $\mathfrak{g}$ in accordance with the rules of acyclic tensor calculus with coefficients in $\mathbb{Q}$. These tensors can conveniently be represented by means of graphs (see the Appendix to [20]). Let U_{rk} be the subspace of V_r corresponding to the connected graphs with first Betti number equal to k. If we choose a basis x_{rki}, $1 \le i \le \dim U_{rk}$, in U_{rk} then $\mathscr{P}_\gamma(\tau)$ can be written as a formal series with rational coefficients in the x_{rki}, where $r = 1, 2, \dots$, $k = 0, 1, \dots$, $1 \le i \le \dim U_{rk}$. If Φ is defined by means of (3.25), the rationality of the coefficients is not obvious, but it follows from the uniqueness part of Theorem 3.15. The author hopes to examine these questions in detail in a later publication.

Now let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra over $\mathbb{C}$ with a fixed invariant scalar product, and t the corresponding element of $\mathfrak{g} \otimes \mathfrak{g}$. From this data can be constructed, on the one hand, a quasitriangular quasi-Hopf QUE-algebra $((U\mathfrak{g})[[h]], \Delta, \Phi, R)$, where Δ is the usual comultiplication, $R = e^{ht/2}$, and Φ is determined as above by means of (3.25), and, on the other hand, a quasitriangular Hopf QUE-algebra $(U_h\mathfrak{g}, \overline{R})$, where $U_h\mathfrak{g}$ has the same meaning as in Example 6.3 of [2], and $\overline{R}$ the same as in §13 of [2].

PROPOSITION 3.16. *$((U\mathfrak{g})[[h]], \Delta, \Phi, R)$ can be turned by a twist into a quasitriangular Hopf* QUE-*algebra isomorphic to* $(U_h\mathfrak{g}, \overline{R})$.

PROOF. As shown in §4 of [21], there exists an algebra isomorphism $\varphi\colon U_h\mathfrak{g} \xrightarrow{\sim} (U\mathfrak{g})[[h]]$, which is the identity $\bmod h$ and such that φ takes the comultiplication $\Delta_h\colon U_h\mathfrak{g} \to U_h\mathfrak{g} \hat{\otimes} U_h\mathfrak{g}$ into a homomorphism $\tilde{\Delta}_h\colon (U\mathfrak{g})[[h]] \to (U\mathfrak{g} \otimes U\mathfrak{g})[[h]]$ of the form $\tilde{\Delta}_h(a) = F^{-1}\Delta(a)F$, $F \in (U\mathfrak{g} \otimes U\mathfrak{g})[[h]]$, $F \equiv 1 \bmod h$. Without loss of generality we can suppose that $(\varepsilon \otimes \mathrm{id})(F) = 1 = (\mathrm{id} \otimes \varepsilon)(F)$. Put $\tilde{R} = (\varphi \otimes \varphi)(\overline{R})$. The quasitriangular Hopf QUE-algebra $((U\mathfrak{g})[[h]], \tilde{\Delta}_h, \tilde{R})$ can be turned, twisting via F, into a quasitriangular quasi-Hopf QUE-algebra of the form $((U\mathfrak{g})[[h]], \Delta, \underline{\Phi}, \underline{R})$. From (1.1) and (3.1) it follows that $\underline{R}$ and $\underline{\Phi}$ are $\mathfrak{g}$-invariant. Furthermore, twisting $((U\mathfrak{g})[[h]], \Delta, \underline{\Phi}, \underline{R})$ via a $\mathfrak{g}$-invariant element of $(U\mathfrak{g} \otimes U\mathfrak{g})[[h]]$, we can make $\underline{R}^{21} = \underline{R}$. The equality $\overline{R}^{21}\overline{R} = \Delta_h(e^{hC_h/2})(e^{-hC_h/2} \otimes e^{-hC_h/2})$, where C_h is the image in $U_h\mathfrak{g}$

of the Casimir element $C \in U\mathfrak{g}$, is proved in §5 of [21]. Therefore $\tilde{R}^{21}\tilde{R} = \tilde{\Delta}_h(e^{hC/2})(e^{-hC/2} \otimes e^{-hC/2})$, so that $\underline{\mathrm{R}}^{21}\underline{\mathrm{R}} = \Delta(e^{hC/2})(e^{-hC/2} \otimes e^{-hC/2}) = e^{ht}$; i.e., $\underline{\mathrm{R}} = e^{ht/2}$. It remains to use the uniqueness part of Theorem 3.15, which is an easy consequence of Proposition 3.11 and formula (3.12). ●

From Proposition 3.16 follows Kohno's theorem [5], which asserts that if ρ is a finite-dimensional representation of $\mathfrak{g}$, and ρ_h the corresponding representation of $U_h\mathfrak{g}$, then the representation of the braid group B_n corresponding to the R-matrix $(\rho_h \otimes \rho_h)(\overline{R})$ is equivalent to the representation of B_n as the monodromy group of the equation obtained from (3.26) by replacing t^{ij} by $\rho^{\otimes n}(t^{ij})$. To be sure, what is asserted in [5] is the equivalence for all $h \notin \pi i\mathbb{Q}$, while what follows from Proposition 3.16 is equivalence for almost all $h \in \mathbb{C}$ (without specification of the "exceptional" set).

BIBLIOGRAPHY

1. L. D. Faddeev, *Integrable models in* $(1+1)$*-dimensional quantum field theory*, Recent Advances in Field Theory and Statistical Mechanics (Les Houches, 1982), North-Holland, Amsterdam and New York, 1984, pp. 561–608.

2. V. G. Drinfel'd, *Quantum groups*, Proc. Internat. Congr. Math. (Berkeley, Calif., 1986), Vol. 1, Amer. Math. Soc., Providence, R. I., 1987, pp. 798–820.

3. N. Yu. Reshetikhin, *Quasitriangular Hopf algebras, solutions of the Yang-Baxter equation, and connection invariants*, Algebra Anal. **1** (1989), no. 2, 30–47; English transl. in Leningrad Math. J. **1** (1990).

4. V. F. R. Jones, *A polynomial invariant for knots via von Neumann algebras*, Bull. Amer. Math. Soc. (N.S.) **12** (1985), 103–111.

5. Toshitake Kohno, *Quantized universal enveloping algebras and monodromy of braid groups*, Preprint, Nagoya Univ., Nagoya, 1988.

6. ——, *Monodromy representations of braid groups and Yang-Baxter equations*, Ann. Inst. Fourier (Grenoble) **37** (1987), 139–160.

7. ——, *Linear representations of braid groups and classical Yang-Baxter equations*, Proc. Conf. on Artin's Braid Groups (Santa Cruz, 1986), Contemporary Math., no. 78, Amer. Math. Soc., Providence, R. I., 1988, pp. 339–363.

8. Akihiro Tsuchiya and Yukihiro Kanie, *Vertex operators in two dimensional conformal field theory on* $\mathbb{P}^1$ *and monodromy representations of braid groups*, Conformal Field Theory and Solvable Lattice Models, Advanced Studies in Pure Math., vol. 16, Academic Press, Boston, Mass., 1988, pp. 297–372.

9. Saunders Mac Lane, *Categories for the working mathematician*, Springer-Verlag, New York and Berlin, 1971.

10. ——, *Natural associativity and commutativity*, Rice Univ. Studies **49** (1963), 28–46.

11. G. M. Kelly, *On Mac Lane's conditions for coherence of natural associativities, commutativities, etc.*, J. Algebra **1** (1964), 397–402.

12. M. E. Sweedler, *Hopf algebras*, Mathematical Lecture Note Series, Benjamin, New York, 1969.

13. D. E. Radford, *A free rank* 4 *Hopf algebra with antipode of order* 4, Proc. Amer. Math. Soc. **30** (1971), 55–58.

14. P. Deligne and J. Milne, *Tannakian categories*, Lecture Notes in Math., vol. 900, Springer-Verlag, Berlin, Heidelberg, and New York, 1982, pp. 101–228.

15. N. Bourbaki, *Groupes et algèbres de Lie*, Éléments de Mathématique, Fasc. 37, Actualités Sci. Indust., No. 1349, Hermann, Paris, 1972.

16. P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory*, Springer-Verlag, New York, 1967.

17. V. G. Drinfel'd, *Constant quasiclassical solutions of the Yang-Baxter quantum equation*, Dokl. Akad. Nauk SSSR **273** (1983), 531–535; English transl. in Soviet Math. Dokl. **28** (1983).

18. J. D. Stasheff, *Homotopy associativity of* H*-spaces*. I, Trans. Amer. Math. Soc. **108** (1963), 275–292.

19. V. G. Knizhnik and A. B. Zamolodchikov, *Current algebra and Wess-Zumino model in two dimensions*, Nuclear Phys. B **247** (1984), 83–103.

20. Roger Penrose and Wolfgang Rindler, *Spinors and space-time*, Vol. 1, Cambridge Univ. Press, Cambridge and New York, 1984.

21. V. G. Drinfel′d, *On almost cocommutative Hopf algebras*, Algebra Anal. **1** (1989), no. 2, 30–46; English transl. in Leningrad Math. J. **1** (1990).

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