POISSON GEOMETRY OF CHARACTER VARIETIES OF SURFACES

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JUNE 27, 2023

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0. Introduction

1. Just enough algebraic geometry

Conventions: The base field is always $\mathbb{C}$. In this section and the next, algebra means an associative, commutative algebra over $\mathbb{C}$.

1.1. **Some classical definitions.** The "naive" definition of an algebraic variety is usually as the set in $\mathbb{C}^n$ of common zeros of a finite set S of polynomials in $\mathbb{C}[x_1, \dots, x_n]$, denoted by

$$Z_S := \{x \in \mathbb{C}^n, \forall f \in S, f(x) = 0\}.$$

This, in particular, induces a topology on Z_S inherited from the complex analytic (i.e. Euclidian) topology on $\mathbb{C}^n$. Let I be the ideal generated by S, then clearly $Z_I = Z_S$. Hilbert's Nullstellensatz [Wik22] states that the ideal in $\mathbb{C}[x_1, \dots, x_n]$ of polynomials vanishing on Z_S is the radical of I

$$\sqrt{I} = \{f \in \mathbb{C}[x_1, \dots, x_n], \exists r \in \mathbb{N}, f^r \in I\}.$$

This establishes a canonical correspondence between algebraic varieties in $\mathbb{C}^n$ and ideals in $\mathbb{C}[x_1, \dots, x_n]$ which are radical, i.e. Satisfies $\sqrt{I} = I$.

The algebra of regular function on $Z(I)$ is defined to be $\mathbb{C}[Z(I)] = \mathbb{C}[x_1 \dots, x_n]/\sqrt{I}$. Let $\mathcal{O}(I)$ be the algebra $\mathbb{C}[x_1, \dots, x_n]/I$. There is a canonical surjective algebra map

$$\mathcal{O}(I) \longrightarrow \mathbb{C}[Z(I)]$$

whose kernel is the nilradical $\sqrt{(0)}$, i.e. the ideal of nilpotent elements in $\mathcal{O}(I)$.

The issue with this definition is that it is not really a definition. Also, the variety at hand a priori depends explicitly on the particular choice of the ideal I. In particular, it's not obvious how to define maps of algebraic varieties (and in particular particular how to decide whether two algebraic varieties are "the same"). So it's actually more natural to go the other way around, starting with the algebra $\mathcal{O} = \mathcal{O}(I)$ one observe that there is a canonical bijection

$$Z_I \cong \mathcal{Z}_{\mathcal{O}}(\mathbb{C}) := \mathrm{Hom}_{\mathrm{alg}}(\mathcal{O}, \mathbb{C})$$

mapping a point $z \in Z_I$ to the evaluation map

$$P \longmapsto P(z).$$

Furthermore, any algebra morphism $\mathcal{O}' \to \mathcal{O}$ induces a map

$$\mathrm{Hom}_{\mathrm{alg}}(\mathcal{O}, \mathbb{C}) \longrightarrow \mathrm{Hom}_{\mathrm{alg}}(\mathcal{O}', \mathbb{C})$$

by precomposition (note that the direction of the map is reversed). One can show that the topology on this space is independent of the chosen presentation, so that it gives a definition of a variety that do not depends on the particular presentation of $\mathcal{O}$. This variety still doesn't quite determine the algebra $\mathcal{O}$ (again, because of the possible existence of nilpotent elements), so the idea is to look at the space of common zeroes for our ideal of polynomial in any commutative $\mathbb{C}$-algebra R, hence we set

$$\mathcal{Z}_{\mathcal{O}}(R) := \mathrm{Hom}_{\mathrm{alg}}(\mathcal{O}, R)$$

and we observe that any algebra map $f : R \to R'$ induces a map $\mathcal{Z}_{\mathcal{O}}(R) \to \mathcal{Z}_{\mathcal{O}}(R')$. We call $\mathcal{Z}_{\mathcal{O}}$ the affine scheme associated with $\mathcal{O}$ and $\mathcal{Z}_{\mathcal{O}}(R)$ its set of R-points. Formally this is really the functor from the category of $\mathbb{C}$-algebras to SET given by $\mathrm{Hom}(\mathcal{O}, -)$. We think of $\mathcal{Z}_{\mathcal{O}}$ as some sort of geometric object of which $\mathcal{O}$ is the algebra of function.

The nice thing is that this actually characterizes $\mathcal{O}$ abstractly in the following sense:

Proposition 1.1. *Let B be an algebra, and assume we're given natural bijections*

$$\mathrm{Hom}_{\mathrm{alg}}(B, R) \cong \mathcal{Z}_{\mathcal{O}}(R)$$

meaning that for any algebra map $f : R \to R'$ the obvious diagram commutes. Then there is a canonical algebra isomorphism

$$\mathcal{O} \xrightarrow{\cong} B.$$

inducing those natural bijections.

In that case this is fairly easy to show directly, but this is a consequence of the so-called Yoneda lemma in category theory.

Remark 1.2. A somewhat more standard definition of an affine scheme is as a pair $(\mathcal{O}, \operatorname{Spec} \mathcal{O})$ where $\mathcal{O}$ is a commutative algebra, and $\operatorname{Spec} \mathcal{O}$ is the set of prime ideals in $\mathcal{O}$, equipped with the so-called Zariski topology. The relation between the two points of view is that an ideal is prime if and only if it is the kernel of a map from $\mathcal{O}$ into a field. In other words, $\operatorname{Spec} \mathcal{O}$ encodes the collections of $\mathcal{Z}_{\mathcal{O}}(R)$ where R runs over all field extensions of $\mathbb{C}$. In particular the set of $\mathbb{C}$ points is identified with the set of maximal ideals.

Hence, abusing terminology somewhat, it will be convenient to make the following definition

Definition 1.3. *A variety Z is the set of $\mathbb{C}$-points of the affine scheme associated with a finitely generated algebra $\mathcal{O}$. We call $\mathcal{O}$ the algebra of function of Z. A morphism of varieties $(Z, \mathcal{O}) \to (Z', \mathcal{O}')$ is just an algebra morphism $\mathcal{O}' \to \mathcal{O}$.*

In other words, every time we'll use the word "variety" we'll always assume that we have a particular algebra associated with it, which will be clear from the context. So at the end of the day this is really just a different way of thinking about the algebra $\mathcal{O}$, so what's the point ? The thing is very often, we know what $\mathcal{Z}_{\mathcal{O}}(R)$ should be from a geometric perspective, and we want to find the associated algebra. The following example will be very useful later in these notes.

Example 1.4. We want to understand $G = \mathrm{GL}_N(\mathbb{C})$ as an affine algebraic variety. We start with the algebra of functions on the space of all square matrices, which we denote by $\mathbb{C}[X]$ where $X = (x_{ij})_{1 \le i,j \le N}$ is a matrix of variables. In other words, if $A = (a_{ij})$ is a matrix, then

$$x_{ij}(A) = a_{ij}.$$

In particular, the determinant of X is an element of this algebra, and we want to impose that it has to be invertible. The trick is to add a variable t and to set

$$\mathcal{O}(\mathrm{GL}_N(\mathbb{C})) := \mathbb{C}[X, t]/(\det(X)t = 1) = \mathbb{C}[X, \det(X)^{-1}]$$

But this is kind of arbitrary, how do we know this is the correct thing ? We could, for example, also introduce a second matrix of variable Y and define it as

$$\mathcal{O}(\mathrm{GL}_N(\mathbb{C})) := \mathbb{C}[X, Y]/(XY - \mathrm{id}).$$

In that case it's easy to show these algebras are isomorphic, but this is a bit tedious and we want something more canonical. Well, if R is an algebra, we definitely know what $\mathrm{GL}_N(R)$ should be: the space of invertible $N \times N$ matrices with coefficients in R, equivalently of matrices whose determinant is invertible in R. But now it's easy to check that for any of those choices we have indeed a natural isomorphism

$$\mathrm{Hom}_{\mathrm{alg}}(\mathcal{O}(\mathrm{GL}_N(\mathbb{C})), R) \cong \mathrm{GL}_N(R)$$

so this really should be taken as the definition of this algebra (this is what we want) and either one of the previous non-canonical constructions shows that this defines something that actually exist.

This point of view also makes it obvious that the multiplication of G is a morphism of variety, In other words that $\mathrm{GL}_N(\mathbb{C})$ is what is called an affine algebraic group (and GL_N is called an affine group scheme). For this, we make the following

Definition 1.5. *Let Z, Z' be two varieties with algebra of functions $\mathcal{O}, \mathcal{O}'$. The product $Z \times Z'$ is the variety whose algebra of function is $\mathcal{O} \otimes \mathcal{O}'$.*

This definition makes sense since we have a bijection

$$\mathrm{Hom}_{\mathrm{alg}}(\mathcal{O}, R) \times \mathrm{Hom}_{\mathrm{alg}}(\mathcal{O}', R) \cong \mathrm{Hom}_{\mathrm{alg}}(\mathcal{O} \otimes \mathcal{O}', R)$$

which is natural in R and given by

$$(\lambda, \lambda') \longmapsto \left(\sum f_i \otimes f_i' \mapsto \sum \lambda(f_i) \lambda'(f_i') \right).$$

Now we have:

Proposition 1.6. *The multiplication*

$$\mathrm{GL}_N(\mathbb{C}) \times \mathrm{GL}_N(\mathbb{C}) \to \mathrm{GL}_N(\mathbb{C})$$

and the inverse map

$$(\ )^{-1} : \mathrm{GL}_N(\mathbb{C}) \to \mathrm{GL}_N(\mathbb{C})$$

are maps of varieties.

Proof. Thanks to the formalism we introduced, it's enough to check that the multiplication

$$\mathrm{GL}_N(R) \times \mathrm{GL}_N(R) \to \mathrm{GL}_N(R)$$

and the inverse map

$$(\ )^{-1} : \mathrm{GL}_N(R) \to \mathrm{GL}_N(R)$$

are natural in R, which is almost tautological. □

1.2. **Quotients.** A good reference for this section is [Bri00]. Let $G = \mathrm{GL}_N(\mathbb{C})$ and Z be a variety with algebra of functions $\mathcal{O}$.

Definition 1.7. *An action of G on Z is said to be algebraic if the action map*

$$G \times Z \longrightarrow Z$$

is a morphism of variety.

Note that, in our setting, an action of G on Z is by definition the same as an action of G on $\mathcal{O}$ by algebra automorphism, but this is not enough for the action to be called algebraic: we also want to take into account the algebraic variety structure on G.

Proposition 1.8. *The action of G on Z is algebraic if and only if for every $f \in \mathcal{O}$, there exists a* finite-dimensional *vector space $V_f \subset \mathcal{O}$ such that $f \in V$ and $G \cdot V = V$.*

Proof. By definition, the action is algebraic if and only if there exists an algebra map

$$\delta : \mathcal{O}(Z) \longrightarrow \mathcal{O}(G \times Z) = \mathcal{O}(G) \otimes \mathcal{O}(Z)$$

such that $\forall g \in G, \forall f \in \mathcal{O}(Z)$

$$g \cdot f = \sum_i h_i(g) f_i$$

where h_i, f_i are defined by

$$\delta(f) = \sum_i h_i \otimes f_i.$$

Since this sum has to be finite, the sub-vector space spanned by the f_i's is finite dimensional. On the other hand, by construction it is stable under the action of G. Conversely, if there exists some finite dimensional G-stable vector space containing f, then it also contains the elements $g \cdot f$ for $g \in G$. Hence the vector space spanned by these elements is again finite dimensional and G-stable. □

In general, the quotient Z/G is bad, in the sense that there is no natural way to put some geometric structure on it. In fact, this is not even Hausdorff in general. Hence we want to define a better quotient. Following again the philosophy we introduced, the question we should ask is "what is a function on Z/G ?". More generally one should wonder what a map out of Z/G is. This should be a map out of Z which is G-invariant in he following sense:

Definition 1.9. *Let Z be a variety with an algebraic action of G. A map of variety*

$$f : Z \longrightarrow W$$

is said to be G-invariant if the corresponding algebra map

$$f^* : \mathcal{O}(W) \to \mathcal{O}(Z)$$

satisfies

$$g \cdot f^* = f^*.$$

Hence we make the following

Definition 1.10. *Let Z be a variety equipped with an action of $G = \mathrm{GL}_N(\mathbb{C})$. A categorical quotient of Z by G is a pair (Y, π) where Y is a variety and*

$$\pi : Z \to Y$$

a map of variety, satisfying the following universal property. For every variety W and any G-invariant map

$$Z \to W$$

there is a unique map

$$Y \to W$$

which makes the following diagram commutes

$$\begin{tikzcd} Z \arrow[r, "\pi"] \arrow[dr] & Y \arrow[d] \\ & W \end{tikzcd}$$

This should holds also if $W = \mathbb{C}$, so a natural thing to do is to define functions on the quotient as G-invariant functions on Z. In other words we can try to define our quotient by

$$\mathcal{O}(Z//G) := \mathcal{O}(Z)^G.$$

where the symbol // means that a priori this is not just the ordinary quotient. First of all we have the (non trivial)

Theorem 1.11 (Hilbert, Nagata)**.** *The algebra $\mathcal{O}(Z//G)$ is finitely generated. Hence $Z//G$ is an affine algebraic variety.*

Then

Proposition 1.12. *The pair $(Z//G, \pi)$ where π is induced by the inclusion*

$$\mathcal{O}(Z)^G \hookrightarrow \mathcal{O}(Z)$$

is a categorical quotient of Z by G.

Proof. By definition, an algebra map $\mathcal{O}(W) \to \mathcal{O}(Z)$ is G-invariant if and only if its image is in $\mathcal{O}(Z)^G$. □

Remark 1.13. One can show (this is non trivial! see [Sch89, Corollary 4.8]) that $Z//G$ is also a categorical quotient in the category of Hausdorff topological space, i.e. any continuous, G-invariant map from Z (seen as a topological space with the complex topology) to some Hausdorff topological space Y factors through a continuous map from $Z//G$, and this space is universal for this property.

The key idea to remember is that since polynomial functions are in particular continuous, they don't see the orbit of the action of G but only their closure. Let us see what happens in two examples.

Example 1.14. Let $\mathcal{O} = \mathbb{C}[X]$ so that $Z = \mathbb{C}$. Let $G = \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$ and $\mathcal{O}(G) = \mathbb{C}[\lambda, \lambda^{-1}]$, and let G acts on Z by multiplication:

$$a \cdot z = az.$$

The ordinary quotient Z/G is made of two points, the orbits $\{0\}$ and $\mathbb{C}^\times$, but the latter is not closed so this space is not Hausdorff. We have no less that three different ways to check that this action is algebraic:

(1) For any algebra R, we have anaction of $\mathrm{GL}_1(R) = R^\times$ on R which is natural in R.
(2) The action corresponds to the algebra map $\mathbb{C}[x] \to \mathbb{C}[X] \otimes \mathbb{C}[\lambda, \lambda^{-1}]$ induced by $x \mapsto x \otimes \lambda$.
(3) The action of $a \in G$ on $\mathbb{C}[x]$ maps a polynomial P to $P(ax)$, hence for any $n \geq 0$ it preserves the finite-dimensional space of polynomials of degree n.

On the other hand, clearly $\mathcal{O}^G = \mathbb{C}$, the constant polynomials, so $Z//G$ is a single point.

Example 1.15. Let $Z = M_N(\mathbb{C})$ on which G acts by conjugation, with the natural choice $\mathcal{O}(Z) = \mathbb{C}[x_{ij}, 1 \leq i, j \leq N]$. Recall that Jordan theorem implies that every matrix can be written as

$$M = M^{ss} + M^{nil}$$

where M^{ss} is diagonalizable and M^{nil} is nilpotent, and M^{ss} and M^{nil} commute. Let

$$M_t = M^{ss} + tM^{nil}, t \in [0, 1].$$

A crucial observation is that if $t \neq 0$, then M_t and M have the same Jordan canonical form so they are conjugated, i.e. they belong to the orbit of M. On the other hand, M_0 is diagonalizable, therefore the closure of the orbit of M contains a diagonalizable matrix.

Conversely, suppose that there exists a continuous family M_t, $t \in]0, 1]$ belonging to the same conjugacy class such that

$$M = \lim_{t \to 0} M_t$$

exists. Our assumptions implies that all the M_t have the same minimal polynomial μ, and because μ is continuous we have

$$\mu(M) = 0.$$

This shows that:

- the closure of the orbit of M actually contains a *unique* orbit of some diagonalizable matrix
- if the M_t are diagonalizable, then so is M (because then μ has no multiplicities, so it has to be the minimal polynomial of M). In other words, the orbit of a diagonalizable matrix is already closed.

Now let P be a G-invariant polynomial on Z, which by definition means that

$$\forall X \in \mathrm{GL}_N(\mathbb{C}), \forall M \in Z, P(X^{-1}MX) = P(X).$$

Then P is constant on every orbit, and because it is continuous it is in fact constant on the closure of every orbit. In particular, it is determined by its value on diagonal matrices. Conversely, if D, D' are two diagonal matrices with different sets of eigenvalues, it's not too hard to construct an invariant polynomial P such that $P(D) \neq P(D')$.

Summing up, we observe the following three facts:

(1) Two matrices M, M' induce the same point in the quotient $Z//G$ if and only if they have the same eigenvalues (counted with multiplicities), if and only if the closure of their orbit have a non-empty intersection.
(2) Since we've shown the closure of an orbit contains a unique closed orbit, the set $Z//G$ can be identified with the set of closed orbits. Hence, let Z^{ss} be the subset of diagonalizable matrices, then $Z//G$ can be identified with the *ordinary* quotient Z^{ss}/G.
(3) the map $\pi : Z \to \mathbb{C}^N/S_N$ which maps a matrices to the unordered set of its eigenvalues induces an isomorphism

$$Z//G \cong \mathbb{C}^N/S_N$$

where again on the right hand side this is the ordinary quotient.

The first two points follow from a general fact about categorical quotients. Note that the algebra of function $\mathcal{O}(\mathbb{C}^N/S_N)$ (which is obviously reduced) is the ring of symmetric polynomials on N-variables, and it is a classical theorem that it is itself a ring of polynomials in N variables. More precisely we have the following classical result, a more general version of which we'll see later:

Theorem 1.16. *Let* Tr^k *be the polynomial function* $Z \to \mathbb{C}$ *defined by* $\mathrm{Tr}^k(M) = \mathrm{Tr}(M^k)$. *Then* Tr^k *is* G*-invariant and there is an isomorphism*

$$\mathcal{O}(Z//G) = \mathcal{O}(Z)^G \cong \mathbb{C}[\mathrm{Tr}, \mathrm{Tr}^2, \ldots, \mathrm{Tr}^N].$$

Now for general categorical quotients we have the following theorem

Theorem 1.17 (Mumford)**.** *Let* $G = \mathrm{GL}_n(\mathbb{C})$ *act algebraically on a variety* Z. *Then:*

(1) The closure of any orbit contains a unique closed orbit

(2) *The map*
$$Z \longrightarrow Z//G$$
is surjective.

(3) $x, y \in Z$ *have the same image iff*
$$\overline{G \cdot x} \cap \overline{G \cdot y} \neq \emptyset.$$

2. Character varieties

2.1. **Representation varieties.** Let Γ be a finitely generated group.

Definition 2.1. *A (finite dimensional) linear representation of* Γ *is a pair* (V, ρ) *of a (finite dimensional) complex vector space* V *and a morphism*
$$\rho : \Gamma \longrightarrow \mathrm{GL}(V).$$
Equivalently this is an action of Γ *on* V *such that the action of each* $g \in \Gamma$ *is given by a linear map.*

We will often denote the action of some $g \in \Gamma$ on some vector $v \in V$ by $g \cdot v$.

Definition 2.2. *A morphism between two representations* (V, ρ), (V', ρ') *is a linear map*
$$\alpha : V \longrightarrow V'$$
such that
$$\forall g \in \Gamma, \forall v \in V, \alpha(g \cdot v) = g \cdot \alpha(v).$$

Two representations ρ, ρ' on the same vector space V are equivalent if there exists $M \in \mathrm{GL}(V)$ such that
$$\forall g \in \Gamma, \rho(g) = M \rho'(g) M^{-1}.$$

Definition 2.3. *The* N*th representation variety* $\mathrm{Rep}(\Gamma)$ *of* Γ *is the set of representations of* Γ *on a fixed* N*-dimensional vector space, i.e. the set of group morphism*
$$\Gamma \longrightarrow \mathrm{GL}_N(\mathbb{C}).$$

Of course it doesn't quite make sense yet to call this a "variety" since we haven't put any geometric structure on it yet. Let's first see an

Example 2.4. A representation of $\mathbb{Z}$ is completely determined by the image of 1, and there are no relation, so we get a bijection
$$\mathrm{Rep}(\mathbb{Z}) \cong \mathrm{GL}_N(\mathbb{C}).$$
Note that this isomorphism depends on our choice of the generator 1.

We've seen in the previous section how to turn $\mathrm{GL}_N(\mathbb{C})$ into a variety. The general idea is very similar: let us first pick a presentation of Γ with finitely many generators:
$$\Gamma = \langle x_1, \dots, x_k |\ r_i(x_1, \dots, x_k), i \in I\rangle$$
where $r_i(x_1, \dots, x_k)$ is some word in the x_a's and their inverses. Then $\mathrm{Rep}(\Gamma)$ can be identified with the subset of G^k of k-tuples of (invertible) matrices
$$M_1, \dots, M_k$$
such that
$$\forall i \in I, r_i(M_1, \dots, M_k) = \mathrm{id}\,.$$
In other words, to define a representation of Γ is the same as picking one matrix for each generator, and to make sure those matrices satisfy the correct relations. Let, as before, X_a be a matrix of variables $X_a = (x^a_{ij})$, then every coefficient of the matrix
$$r_i(X_1, \dots, X_n) - \mathrm{id}$$
is a polynomial in the coefficients of the matrices X_a and of their inverse (note that the coefficients of the inverse of X_a are themselves polynomial in the coefficients of X_a and of $\det(X_a)^{-1}$). In other words, each coefficient of
$$r_i(X_1, \dots, X_n) - \mathrm{id}$$

is a polynomial function on G^k and $\mathrm{Rep}(\Gamma)$ is the set of common zeros of those polynomials in G^k. Hence we define the algebra $\mathcal{O}(\mathrm{Rep}(\Gamma))$ as the quotient of $\mathcal{O}(G^k)$ by the ideal generated by these polynomials. We want to show that none of that depends on the chosen presentation of Γ.

Proposition 2.5. *The algebra $\mathcal{O} = \mathcal{O}(\mathrm{Rep}(\Gamma))$ satisfies the following universal property: for any $\mathbb{C}$-algebra R, there is a canonical, natural bijection*

$$\mathcal{Z}_{\mathcal{O}}(R) = \mathrm{Hom}_{\mathrm{alg}}(\mathcal{O}, R) \cong \{\rho : \Gamma \longrightarrow \mathrm{GL}_N(R)\}.$$

Proof. Set $\mathcal{O}(G^k) = \mathbb{C}[X_1, \ldots, X_k, \det(X_1)^{-1}, \ldots, \det(X_k)^{-1}]$. By definition, the entries (x^a_{ij}) of X_a are elements of this algebra, so we can think of X_a as an actual matrix with coefficients in this algebra, i.e. as an element of $\mathrm{GL}_N(\mathcal{O}(G^k))$, and thus we can take its image in $\mathrm{GL}_N(\mathcal{O}(\mathrm{Rep}(\Gamma)))$. By construction there is a morphism

$$\Gamma \longrightarrow \mathrm{GL}_N(\mathcal{O}(\mathrm{Rep}(\Gamma)))$$

which maps the generator x_a to the matrix X_a.

Therefore, given any representation

$$\Gamma \longrightarrow \mathrm{GL}_N(R)$$

for some algebra R, we define a map

$$\mathcal{O}(\mathrm{Rep}(\Gamma)) \to R$$

by sending each generator x^a_{ij} to m^a_{ij} where (m^a_{ij}) are the entries of $M_a = \rho(x_a)$. Conversely, given any algebra morphism

$$f : \mathcal{O}(\mathrm{Rep}(\Gamma)) \longrightarrow R$$

we construct a representation as the composition

$$\Gamma \to \mathrm{GL}_N(\mathcal{O}(\mathrm{Rep}(\Gamma))) \to \mathrm{GL}_N(R)$$

where the second map is induced by f. Those maps are clearly inverse to each other, which prove the universal property of $\mathcal{O}(\mathrm{Rep}(\Gamma))$. $\square$

Warning 2.6. This algebra is *not* always reduced, it can contains nilpotent elements.

2.2. **Character varieties and characters.** The group $G = \mathrm{GL}_N(\mathbb{C})$ acts on $\mathrm{Rep}(\Gamma)$ by conjugation (i.e. two representations are in the same orbit iff they are equivalent). In fact, we have an action

$$\mathrm{GL}_N(R) \times \mathrm{Rep}(\Gamma)(R) \to \mathrm{Rep}(\Gamma)(R)$$

naturally in R, hence this action of G is algebraic in the sense of Section 1.2. We can thus make the following definition:

Definition 2.7. *The* character variety *of Γ is the categorical quotient*

$$\mathrm{Ch}(\Gamma) := \mathrm{Rep}(\Gamma) /\!/ \mathrm{GL}_N(\mathbb{C}).$$

In this section we want to describe the points of this quotient.

Definition 2.8.
- *A sub-representation of V is a subspace $W \in V$ such that $\Gamma \cdot W \subset W$.*
- *A representation V is simple (or irreducible) if it is nonzero and the only subrepresentations are V and (0).*
- *A representation V is semisimple if for every sub-representation W there exists another sub-representation W' such that*

$$V = W \oplus W'.$$

Remark 2.9. It follows from the definition that every semisimple representation can be written as a direct sum

$$W = \bigoplus W_i$$

of representations where each W_i is irreducible. There is however no *canonical* way to do this in general.

Definition 2.10. *Let (V,ρ) be a representation of Γ. Its character is the function*

$$\chi_\rho : \Gamma \longrightarrow \mathbb{C}$$

defined by

$$\gamma \longmapsto \mathrm{Tr}(\rho(\gamma)).$$

Recall that if M, N are two matrices, then

$$\mathrm{Tr}(MN) = \mathrm{Tr}(NM).$$

It implies the trace is invariant under *cyclic* permutations, and in particular that it's invariant under conjugation since

$$\mathrm{Tr}(MNM^{-1}) = \mathrm{Tr}(M^{-1}MN) = \mathrm{Tr}(N).$$

This property of the trace is of course well-known, but it will play such a fundamental role in these notes that it's probably worth emphasizing it. In the case at hand it implies the following Proposition.

Proposition 2.11. *Two equivalent representations of Γ define the same character.*

Proof. If two representations ρ, ρ' are conjugated by say some element $M \in \mathrm{GL}(V)$ then

$$\forall g \in \Gamma, \chi_{\rho'}(g) = \mathrm{Tr}(M\rho(g)M^{-1}) = \mathrm{Tr}(\rho(g)) = \chi_\rho(g).$$

□

The converse is however not true: if V is a representation and $U \subset V$ a subrepresentation, and set $W = U/V$. Then, choosing a basis for V and completing to get a basis of U, the representation on U can be written as

$$\begin{pmatrix} \rho_U & * \\ 0 & \rho_W. \end{pmatrix}$$

Taking the trace it follows that $\chi_V = \chi_U + \chi_W$ even if U is not isomorphic to $U \oplus V$. In other words, the character cannot tell if a representation is semisimple or not. We'll see more on this later.

Either way, this shows that the point in the character variety defined by a representation depends only on its character. The main goal of this section is to prove a converse to this statement, which justifies the name "character variety":

Theorem 2.12 (Artin-Voigt)**.** *There are canonical bijections between the following sets:*

(1) the character variety $\mathrm{Ch}(\Gamma)$
(2) the set of characters of N-dimensional representations of Γ
(3) the set of equivalence classes of semisimple *N-dimensional representations of Γ.*

Again, to get a feeling of what's going on, let us start with an example.

Example 2.13. We've seen that a representation of $\mathbb{Z}$ is determined by the image of 1 which is some invertible matrix M. We claim that:

(1) A representation of $\mathbb{Z}$ is irreducible iff it is 1-dimensional.
(2) Two representations are equivalent iff the matrices images of 1 are conjugated, so that up to equivalence representations are classified by Jordan normal forms.
(3) Hence, a representation is semisimple iff the matrix image of 1 is diagonalizable.

Therefore, we see that the points of $\mathrm{Ch}(\mathbb{Z})$ are indeed identified with equivalence classes of *semisimple* representations.

2.3. Proof of Theorem 2.12.

Definition 2.14. *The group algebra $\mathbb{C}[\Gamma]$ of Γ is the algebra defined as follows: as a $\mathbb{C}$-vector space it is generated by Γ, and the multiplication is induced by the multiplication of Γ*

$$\left(\sum_{g\in\Gamma} \lambda_g g\right)\left(\sum_{h\in\Gamma} \mu_h h\right) := \sum_{g,h\in\Gamma} \lambda_g \mu_h gh.$$

The point is that a representation of Γ is exactly the same as a module over $\mathbb{C}[\Gamma]$. In what follows all modules are assumed to be finite-dimensional (as vector spaces over $\mathbb{C}$). We will say that a module over an arbitrary, non-necessarily commutative algebra A over $\mathbb{C}$ is simple if it doesn't have any proper submodule, and semi-simple if it can be written as a direct sum of simple modules. An A-module structure on some finite dimensional vector space M is the same as an algebra map $\rho : A \to \mathrm{End}(M)$, and we define the character χ_M of M as the linear map

$$a \longmapsto \mathrm{Tr}(\rho(a)).$$

The following Lemma is fundamental in representation theory.

Lemma 2.15 (Schur). *Let M, N be simple A-modules.*

(1) Every nonzero morphism of A-module $M \to N$ is an isomorphism (in particular, if M and N are not isomorphic, then $\mathrm{Hom}_A(M, N) = (0)$).
(2) $\mathrm{End}_A(M) = \mathbb{C}$. In other words, if M and N are isomorphic then this isomorphism is unique up to multiplication by a constant.

Proof. (1) Let $f : M \to N$ be a nonzero A-module morphism. Then the kernel of f is a submodule of M which is not the whole of M since f is nonzero. Since M is simple, it means that $\ker f = (0)$. Likewise, the image of f is a submodule of N which cannot be (0), so it has to be N since it is simple.

(2) Let $f : M \to N$ be a nonzero A-module morphism. Since $\mathbb{C}$ is algebraically closed (note this wasn't required for the previous point), f has a nonzero eigenvalue λ. Let v be a nonzero eigenvector with eigenvalue λ. Then, $f - \lambda\,\mathrm{id}$ is again an A-module map, and its kernel is nonzero since it contains v. Hence, its kernel has to be the whole of M, therefore $f = \lambda\,\mathrm{id}$.

□

Let now M be an irreducible A-module with action given by $\rho : A \to \mathrm{End}(M)$. Then $\mathrm{End}(M)$ is also an A-module where A acts as

$$a \cdot f := \rho(a) \times f.$$

This module is semisimple: if $(e_1, \dots, e_n)$ is a basis of M (as a vector space) then the map

$$f \longmapsto (f(e_1), \dots, f(e_n))$$

is an isomorphism of A-module. Note that the same is true for any module, but since M is simple we cannot decompose $\mathrm{End}(M)$ further.

Theorem 2.16 (Density theorem). *Let $M_1, \dots, M_r$ be simple A-modules which are pairwise non-isomorphic. Then the algebra map*

$$A \longrightarrow \bigoplus \mathrm{End}(M_i)$$

is surjective.

Proof. We first prove the statement for $r = 1$. Set $M = M_1$ and let $m_1, \dots, m_k$ be a set of linearly independent elements of M. We claim that for any other $(n_1, \dots, n_k)$ there exists $a \in A$ such that $n_i = a \cdot m_i$. In particular, for $k = \dim(M)$, i.e. if the m_i's form a basis, this imply the statement of the Theorem: if $f \in \mathrm{End}(M)$, the claim implies that there exists $a \in A$ such that

$$\forall i = 1 \dots k, f(n_i) = a \cdot m_i.$$

But since a linear map $M \to M$ is determined by the image of some basis this shows the image of a in $\mathrm{End}(M)$ is f, hence this map is surjective.

We prove he claim by induction: if $k = 1$, then the condition just say that $m_1 \neq 0$, and since M is simple, $A \cdot m_1 = M$.

Now in the general case, we claim there exists $a \in A$ such that

$$a \cdot m_1 = a \cdot m_2 \cdots = a \cdot m_{k-1} = 0$$

and

$$a \cdot m_k \neq 0.$$

If that's no true, then
$$\forall a, a\cdot m_1 = a\cdot m_2 \cdots = a\cdot m_{k-1} = 0 \Rightarrow a\cdot m_k = 0$$
Since $(m_1,\dots,m_k)$ generates a copy of M^{k-1} inside $\mathrm{End}(M) = M^{\dim(M)}$, by induction hypothesis, this shows there is a well defined A-module map
$$M^{k-1} \to M$$
given by
$$(a\cdot m_1,\dots,a\cdot m_{k-1}) \mapsto a\cdot m_k.$$
By Schur Lemma, this sends $(m_1,\dots,m_{k-1})$ to $\sum \lambda_i m_i$ for some $\lambda_i \in \mathbb{C}$ which contradicts the fact that the m_i's are linearly independent.

Hence, for $1 \leq i \leq k$ we can choose a_i such that $a_i\cdot m_i \neq 0$ and $a_i\cdot m_j = 0$ for $i \neq j$, and we can also choose b_i such that $n_i = b_i a_i m_i$. Then setting $a = \sum b_i a_i$ we get that $n_i = a m_i$ as required.

For the case of general r, let B be the image of A in $\bigoplus \mathrm{End}(M_i)$ and let B_i be the image of A in $\mathrm{End}(M_i)$. Schur Lemma implies that $B = \bigoplus B_i$, and the previous point that $B_i = \mathrm{End}(M_i)$.

□

Corollary 2.17. *Let $M_1,\dots,M_r$ be simple (finite-dimensional) A-modules which are pairwise non-isomorphic. Then the characters of these modules are linearly independent (as elements of $\mathrm{Hom}_{\mathbb{C}}(A,\mathbb{C})$).*

Proof. The density Theorem implies the map
$$A \to \bigoplus \mathrm{End}(M_i)$$
is surjective. Let χ_i be the character of M_i and suppose that there exists a linear relation
$$\sum \lambda_i \chi_i = 0$$
with the λ_i's are not all zero. It implies that for any choice of $f_1,\dots,f_r \in \bigoplus \mathrm{End}(M_i)$
$$\sum \lambda_i \operatorname{Tr}(f_i) = 0$$
which is clearly impossible. □

Definition 2.18. *Let M be a finite dimensional A-module. A composition series is a strictly increasing series of submodules*
$$M_0 = (0) \subset M_1 \subset \cdots \subset M_n = M$$
such that $N_i = M_{i+1}/M_i$ is simple.

Theorem 2.19 (Jordan–Hölder)**.** *Any two composition series have the same length, and the set of associated simple modules, counted with multiplicities, are the same.*

Proof. Let $S_1,\dots,S_k$ be the set of pairwise distinct simple modules associated with a composition series of M and let k_i its multiplicity (the number of time it appears). Then
$$\chi_M = \sum k_i \chi_{S_i}.$$
By Corollary 2.17, the isomorphism class of each S_i and the integers k_i are completely determined by χ_M, and in particular do not depend on the choice of the compositions series. □

Definition 2.20. *We call the semi-simple module*
$$\bigoplus N_i$$
the semisimplifcation of M. This is well-defined up to non-canonical equivalence thanks to the Jordan–Hölder theorem.

Corollary 2.21. *Two A-modules M, M' have the same character if and only if they have the same semisimplification. In particular, there is a canonical bijection between the set of N-dimensional characters, and the set of N-dimensional semi-simple representations.*

Now we come back to representations of Γ[1]. Recall that we fixed $G = \mathrm{GL}_N(\mathbb{C})$ and $V = \mathbb{C}^N$.

[1] of course character varieties can be defined for arbitrary algebras, we just haven't done it.

Lemma 2.22. *Let $\rho \in \operatorname{Rep}(\Gamma)$ and $O = G \cdot \rho$ its orbit. Then the closure $\overline{O}$ of O contains the semisimplification of ρ.*

Proof. This, of course, should be compared to Example 1.15. Let $V_1 \subset V$ be a simple subrepresentation. We've already seen that ρ can be written as

$$\begin{pmatrix} \rho_{V_1} & A \\ 0 & \rho_V/V_1. \end{pmatrix}$$

Conjugating this matrix by $t\,\mathrm{id}_{V_1} \oplus \mathrm{id}\, V/V_1$ for some $t \neq 0$ one gets

$$\begin{pmatrix} \rho_{V_1} & tA \\ 0 & \rho_{V/V_1}. \end{pmatrix}$$

Iterating this process, we obtain that ρ is conjugated to a representation ρ_t of the form

$$\begin{pmatrix} \rho_{V_1} & & & \\ & \rho_{V_2} & B_t & \\ & & \ddots & \\ 0 & & & \rho_{V_n}. \end{pmatrix}$$

where B_t is some upper triangular matrix whose coefficients are polynomials in t and such that $B_0 = 0$. By construction, the V_i's are precisely the simple module appearing in a composition series for V. Hence, $\lim_{t\to 0} \rho_t$ is precisely the semisimplification of V. □

Hence, to finish the proof of Theorem 2.12, it remains to show that if ρ, ρ' are two non-equivalent semisimple representations, then they induce different points in $\operatorname{Ch}(\Gamma)$. Thanks to Mumford's Theorem 1.17, this also implies that orbits of semi-simple representations are closed[2] .

Observe that any $g \in \Gamma$ induces an invariant function f_g on $\operatorname{Rep}(\Gamma)$ which maps a representation ρ to $\chi_\rho(g)$. Choose a presentation of Γ with generators $x_1, \dots, x_n$. This gives a presentation of $\mathcal{O}(\operatorname{Rep}(\Gamma))$ as a quotient of $\mathbb{C}[X_1, \dots X_n]$, and writing g as a words in the $x_i's$ and their inverse we can construct a matrix of polynomials X_g. Then, we claim that f_g is in fact a polynomial function, i.e. it belongs to $\mathcal{O}(\operatorname{Ch}(\Gamma))$ since it is given by

$$f_g := \operatorname{Tr}(X_g).$$

Since ρ, ρ' are not equivalent, their characters $\chi_\rho, \chi_{\rho'}$ are different. Let $g \in \Gamma$ be such that

$$\chi_\rho(g) \neq \chi_{\rho'}(g).$$

Then $f_g(\rho) \neq f_g(\rho')$ as required.

3. Character varieties of surfaces

3.1. Character varieties of topological manifolds.

Let X be a path connected "reasonable" topological manifold whose fundamental group is finitely generated, and $x \in X$. We set

$$\operatorname{Rep}(X, x) := \operatorname{Rep}(\pi_1(X, x)) \qquad\qquad \operatorname{Ch}(X, x) := \operatorname{Ch}(\pi_1(X, x)).$$

Proposition 3.1. $\operatorname{Ch}(X, x)$ *is* canonically *independent of the basepoint x.*

Proof. Let $y \in X$ be another basepoint and γ be a path from x to y. For any commutative algebra R γ induces a natural bijection

$$\operatorname{Rep}(\pi_1(X, x))(R) \longrightarrow \operatorname{Rep}(\pi_1(X, y)(R)$$

mapping ρ to the representation ρ_γ defined by[3]

$$\Pi_1(X, y) \ni \mu \longmapsto \rho(\gamma\mu\gamma^{-1}).$$

This induces an isomorphism of varieties $\operatorname{Rep}(X, x) \cong \operatorname{Rep}(X, y)$, which in general depends on the choice of (the homotopy class of) γ. If γ is another path from x to y, then $\gamma'\gamma^{-1}$ is a loop based at y and ρ_γ and $\rho_{\gamma'}$

[2]In the literature one usually proves directly that these are closed, and then uses Mumford theorem to deduce the statement we want. However the proof of this is rather cumbersome, so we use this trick instead.

[3]We'll use the same symbol for a path and its homotopy class, and write composition from right to left.

are conjugated by $\rho(\gamma'\gamma^{-1})$. Therefore, ρ_γ and $\rho_{\gamma'}$ induces the same point in $\mathrm{Ch}(X, y)(R)$. In other words, the induced isomorphism of varieties

$$\mathrm{Ch}(X, x) \cong \mathrm{Ch}(X, y)$$

do not depends on this choice. $\square$

Thanks to this Proposition, from now on we'll write $\mathrm{Ch}(X)$ for "the" character variety of X.

Proposition 3.2. *Let X, Y be two topological manifolds and $f : X \to Y$ a continuous map. For any basepoint $x \in X$, f induces morphism of algebraic variety*

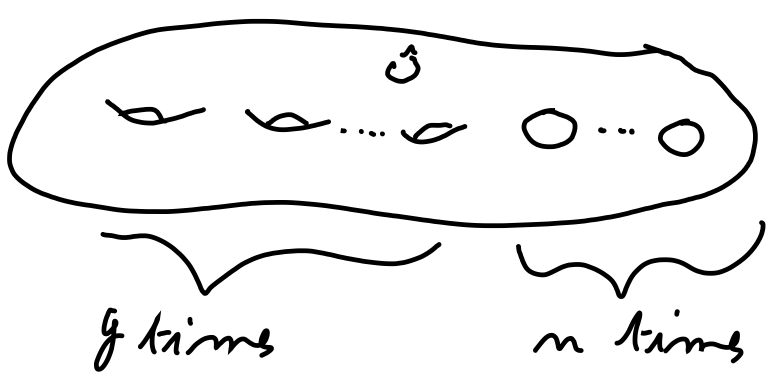
$$\mathrm{Rep}(Y, f(x)) \longrightarrow \mathrm{Rep}(X, x)$$

which descends to a canonical morphism

$$\mathrm{Ch}(Y) \longrightarrow \mathrm{Ch}(X).$$

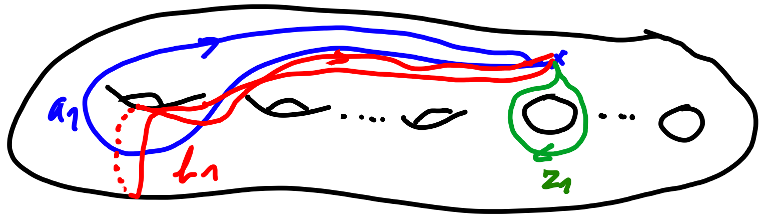
Proof. Clear. $\square$

3.2. **Fundamental groups of surfaces.** For us "surface" means "compact connected oriented 2-dimensional manifold, maybe with boundaries". Up to diffeomorphisms, surface are classified by pairs (g, n) of non-negative integers: the genus g, i.e. the number of handles, and the number n of boundary components. In other words, every surface is diffeomorphic to one that looks like this:



We will denote this "standard" surface, with orientation chosen as shown on the picture, by $S_{g,n}$, and we set $S_g = S_{g,0}$.

Choose a basepoint on $S_{g,n}$ and define loops a_i, b_i, z_j, $1 \le i \le g, 1 \le j \le n$ as on this picture:



Theorem 3.3. *The fundamental group of $S_{g,n}$ is generated by (a_i, b_i, z_j) with a single relation*

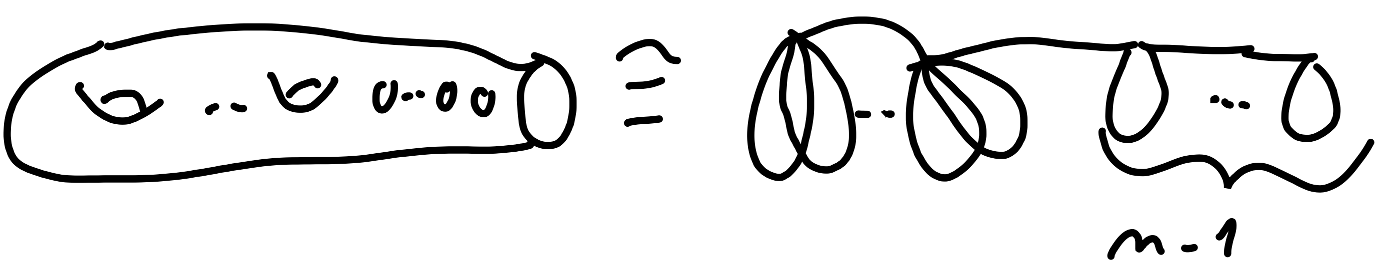
$$\prod_i (a_i, b_i) \prod_j z_j = 1$$

where

$$(a_i, b_i) = a_i b_i a_i^{-1} b_i^{-1}$$

is the commutator.

In particular, if $n > 0$, then this single relation allows one to write z_n as a product of the other generators and one can show that the fundamental group in that case is isomorphic to the free group on the $2g + n - 1$ generators (a_i, b_i, z_j), $i = 1 \dots n$, $j = 1 \dots n - 1$, with the convention that if $g = 0$ (resp. $n = 1$) there are no generators a_i, b_i (resp. z_j). This can be shown, e.g., by observing that $S_{g,n}$ is homotopy equivalent to a wedge of circles:



The case of closed surfaces is trickier, see e.g. [Hat05, Section 1.2]. The first non-trivial example in the closed case is the torus which has fundamental group isomorphic to $\mathbb{Z}^2$.

3.3. **Character varieties of surfaces.** If $n > 0$, this choice of generators induces an isomorphism

$$\mathrm{Ch}(S_{g,n}) \cong \mathrm{GL}_N(\mathbb{C})^{2g+n-1} /\!/ \mathrm{GL}_N(\mathbb{C}).$$

Then, $\mathrm{Ch}(S_g)$ can be described as follows: Let $\mathbb{A}$ be the annulus and let

$$f : \mathbb{A} \longrightarrow S_{g,1}$$

be the inclusion around the boundary (we assume we have chosen a compatible basepoint on each which we'll suppress from the notations) as in the following picture:

Then the induced map

$$f_* : \mathrm{Rep}(S_{g,1}) = G^{2g} \longrightarrow \mathrm{Rep}(\mathbb{A}) = G$$

maps a tuple of matrices $(A_1, \dots, A_g, B_1, \dots, B_g)$ to $\prod(A_g, B_g)$, and

$$\mathrm{Ch}(S_g) = f_*^{-1}(\{\mathrm{id}\}) /\!/ G.$$

Let I be the so-called augmentation ideal of $\mathcal{O}(G)$, i.e. the ideal of polynomial functions P such that $P(\mathrm{id}) = 0$. Let

$$\tilde{f} : \mathcal{O}(G) \to \mathcal{O}(G^{2g})$$

be the algebra map induced by f. Then

$$\mathcal{O}(\mathrm{Ch}(S_g)) = \mathcal{O}(G^{2g})/(\tilde{f}(I)).$$

4. The Goldman algebra

We saw in the proof of Theorem 2.12 that every element γ of the fundamental group of $\pi_1(S, x)$ induces an element P_γ in $\mathcal{O}(\mathrm{Ch}(S))$. At the level of $\mathbb{C}$-points this is the polynomial function which maps a character χ to $\chi(\gamma)$. We will prove that this element do not depends on γ but only on the associated "free loop" on S obtained from γ by forgetting the basepoint. In this section, we'll define a certain algebra whose elements are linear combination of unions of loops on S and explain that this is some sort of universal version of algebras of functions on character varieties of S for all $N \geq 1$, equivalently of polynomials in certain trace functions on the fundamental group of S.

Definition 4.1. *Let X be a topological manifold. The free loop space of X is the topological space*

$$LX := Map(S^1, X)$$

of continuous maps from the circle S^1 to X, equipped with the compact open topology. An homotopy between two free loops $\hat{\gamma}, \hat{\mu}$ is a continuous map

$$h : S^1 \times [0,1] \longrightarrow X$$

such that $h(0, -) = \hat{\gamma}$ and $h(1, -) = \hat{\mu}$.

Definition 4.2. *The* Goldman module *of X is the $\mathbb{C}$-vector space generated by homotopy classes of free loops on X*

$$\mathcal{L}(S) = \mathbb{C}[\pi_0(LX)].$$

In other words, this vector space is made of formal $\mathbb{C}$-linear combinations

$$\sum_{\hat{\gamma} \in \pi_0(LX)} \lambda_{\hat{\gamma}} \hat{\gamma}, \ \ \lambda_{\hat{\gamma}} \in \mathbb{C}$$

where only finitely many $\lambda_{\hat{\gamma}}$ are nonzero.

Remark 4.3. It is a standard fact in algebraic topology that this is the same as the 0th homology group of LX with coefficients in $\mathbb{C}$

$$\mathcal{L}(S) = H_0(LX, \mathbb{C}).$$

To a large extent the structures we're considering in theses notes are the shadows of much richer structures on the full homology of LX.

Proposition 4.4. *Let $\hat{\pi}(X)$ be the set of conjugacy classes in $\pi_1(X,x)$. The map*

$$\pi_1(X,x) \longrightarrow \pi_0(LX)$$

given by "forgetting the basepoint" induces a bijection

$$\hat{\pi}(X) \cong \pi_0(LX).$$

Proof. We first observe that this map is well-defined: if two loops γ, γ' represent elements of the fundamental group which are conjugated by an element represented by a loop μ, then we can apply an homotopy to γ that moves the basepoint along μ and get γ', so they're homotopic as free loops.

Every free loop on X is homotopic to one that contains the chosen basepoint x, so this map is surjective. Let γ, γ' be two based loops which have the same image through this map, which means that there is an homotopy between them which might however not preserve the basepoint. This homotopy will "move the basepoint and put it back", so let μ be the path followed by the base point during this process. We see that γ and γ' are related by conjugation by μ, i.e. they are in the same conjugacy class. □

Definition 4.5. *Let W be a (possibly infinite-dimensional) vector space over $\mathbb{C}$. The* symmetric algebra *$S(W)$ is the free commutative algebra generated by W. If $(e_i)_{i\in I}$ is a basis of W, then this is the algebra of polynomials in the variables e_i:*

$$S(W) = \mathbb{C}[e_i \mid i \in I].$$

Remark 4.6. This is not to be confused with the algebra $\mathcal{O}(W)$ of polynomial function *on* W. Those two algebras are in some sense dual since we claim that

$$\mathcal{O}(W) = S(W^*).$$

In particular, if W is finite-dimensional then they're isomorphic but very non-canonically and it's often important to distinguish between them even in that case.

Definition 4.7. *The Goldman algebra is defined by*

$$\mathcal{G}(X) := S(\mathcal{L}(X)).$$

Remark 4.8. The Goldman algebra can also be described as follows:

- elements are given by formal linear combinations of "free multi-loops", i.e. union of finitely many free loops on X
- multiplication is given by the union of loops
- the unit is the empty loop.

Proposition 4.9. *There is a unique algebra map*

$$\mathrm{ev} : \mathcal{G}(X) \longrightarrow \mathcal{O}(\mathrm{Ch}(X))$$

which maps a free loop $\hat{\gamma}$ to the element obtained by picking a basepoint and a path from that basepoint to $\hat{\gamma}$, obtaining this way a based loop γ, and mapping $\hat{\gamma}$ to the function P_γ.

Proof. Any choice of a basepoint x gives a canonical isomorphism

$$\mathcal{O}(\mathrm{Ch}(S)) \cong \mathcal{O}(\mathrm{Ch}(\pi_1(S,x)))$$

so we can assume we have fixed this basepoint once and for all. Proposition 4.4 shows that for any two homotopic free loops $\hat{\gamma}, \hat{\gamma}'$, and for any choice of a path from x to each of those loops, the corresponding lifts in $\pi_1(S,x)$ are conjugated. It follows that the elements in $\mathcal{O}(\mathrm{Ch}(S))$ associated with these lifts are equal. We obtain this way a map

$$\pi_0(LX) \to \mathcal{O}(\mathrm{Ch}(S))$$

which extends uniquely to an algebra map

$$\mathcal{G}(S) \to \mathcal{O}(\mathrm{Ch}(S))$$

□

We will see in Section 6.1 that this map is in fact surjective and we'll compute its kernel. Let us see a simple example in the meantime.

Example 4.10. Recall that

$$\mathcal{O}(\mathrm{Ch}(\mathbb{A})) \cong \mathcal{O}(G)^G \cong \mathbb{C}[\mathrm{Tr}_1, \dots, \mathrm{Tr}_N, \det{}^{-1}].$$

Part of this statement implies that det is a polynomial in $\mathrm{Tr}_1, \dots, \mathrm{Tr}_N$. But since $\det(A)^{-1} = \det(A^{-1})$ this means that $\det^{-1}$ can also be written as a polynomial in $\mathrm{Tr}_{-1}, \dots, \mathrm{Tr}_{-N}$. In other words this algebra is also generated by $\mathrm{Tr}_k, k = -N \dots N$ with some relations.

The Goldman algebra of the annulus is the algebra

$$\mathbb{C}[t_k, k \in \mathbb{Z}]$$

where t_k represents a loop turning around the "hole" k times counterclockwise if $k > 0$ and clockwise if $k < 0$. One should then think of t_k as a universal version of Tr_k, i.e. as the function that maps an invertible matrix A of *any size* to $\mathrm{Tr}(A^k)$. In particular, for any $N \geq 1$ the map of Proposition 4.9

$$\mathcal{G}(\mathbb{A}) \to \mathcal{O}(\mathrm{GL}_N(\mathbb{C}))^{\mathrm{GL}_N(\mathbb{C}))}$$

sends t_k to Tr_k.

Definition 4.11. *Let A be a (non-necessarily commutative) algebra. The trace $\mathrm{Tr}(A)$ of A is the quotient of A by the vector space (not the ideal !) generated by the commutators $[a, b] = ab - ba$ for all $a, b \in A$.*

A trace on A is a linear map t from A to $\mathbb{C}$ such that

$$\forall a, b \in A, t(ab) = t(ba).$$

Thus, a trace on A is exactly the same as a linear map $\mathrm{Tr}(A) \to \mathbb{C}$, hence the name.

Let Γ be a group. It follows directly from the definition that a linear map $\mathbb{C}[\Gamma] \to \mathbb{C}$ is a trace if and only if it is constant on conjugacy classes. In the particular case at hand, this implies that there are natural linear isomorphisms

$$\mathrm{Tr}(\mathbb{C}[\pi_1(X), x)]) \cong \mathbb{C}[\widehat{\pi}_1(S, x)] \cong \mathcal{L}(S).$$

This provides us with a third interpretation of the Goldman algebra.

5. The Poisson structure

5.1. **Poisson algebras.** In this section we introduce the notions of Lie algebra and Poisson algebra. For an excellent reference on this topic, see [LGPV12].

Definition 5.1. *A Lie algebra is a vector space L together with a bilinear map (the "bracket")*

$$[\ ,\ ] : L \times L \longrightarrow L$$

satisfying the following axioms:

- *antisymmetry*

$$[x, y] = [-y, x]$$

- *The Jacobi identity*

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

Example 5.2. (1) In general, if A is an associative algebra, then the commutator

$$[a, b] = ab - ba$$

turns A into a Lie algebra.

(2) An important particular case of this is the Lie algebra $\mathfrak{gl}_n$, the space of square $n \times n$ matrices quipped with the commutator.

Definition 5.3. *Let $\mathcal{O}$ be a commutative algebra. A* derivation *of $\mathcal{O}$ is a linear map $d : \mathcal{O} \to \mathcal{O}$ which satisfies the Leibniz identity*

$$d(ab) = d(b)a + d(a)b.$$

Example 5.4. The linear map $\frac{\partial}{\partial x}$ is a derivation of the algebra $\mathcal{O} = \mathbb{C}[x]$.

Definition 5.5. *Let $\mathcal{O}$ b a commutative algebra. A Poisson bracket on $\mathcal{O}$ is a bilinear map*

$$\{\ ,\ \} : \mathcal{O} \times \mathcal{O} \longrightarrow \mathcal{O}$$

sch that:

(1) $(\mathcal{O}, \{\ ,\ \})$ *is a Lie algebra*
(2) *for any $x \in \mathcal{O}$, the linear maps*

$$y \longmapsto \{x, y\}$$

and

$$y \longmapsto \{y, x\}$$

are derivations of $\mathcal{O}$.

Remark 5.6. Because of the anti-symmetry relations, each one the above linear maps is a derivation if and only if the other one is.

Exercise 5.7. Show that a Poisson bracket is completely determined by its value on a set of generators of $\mathcal{O}$.

Example 5.8. The fundamental example of a Poisson bracket, which originally arises in classical mechanics, is on $\mathbb{C}[x, p]$ where x should be thought of as the position of some particle and p is its "momentum". It is characterized by the so-called "canonical relation"

$$\{x, p\} = 1.$$

More generally, for $f, g \in \mathbb{C}[x, p]$

$$\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial x}.$$

The theory is then formulated using the Hamiltonian, an element $H \in \mathbb{C}[x, p]$ and the time evolution of some $f_t \in \mathbb{C}[x, p, t]$ is given by

$$\frac{\partial f}{\partial t} = -\{H, f_t\}.$$

For example, a simple harmonic oscillator, i.e. some object of mass m attached to a spring with 'spring constant" k is given by

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2.$$

Example 5.9. Let $(\mathfrak{g}, [\ ,\ ])$ be a Lie algebra. There is a unique Poisson bracket on $S(\mathfrak{g})$, the Kirilov–Kostant–Souriau bracket, characterized by

$$\forall a, b \in \mathfrak{g} \subset S(\mathfrak{g}), \{a, b\} = [a, b].$$

Exercise 5.10. Let $e_i, i \in I$ be a basis of $\mathfrak{g}$ so that

$$S(\mathfrak{g}) \cong \mathbb{C}[e_i, i \in I].$$

Show that for $f, g \in S(\mathfrak{g})$

$$\{f, g\} = \sum_{i,j \in I} [e_i, e_j] \frac{\partial f}{\partial e_i}\frac{\partial g}{\partial e_j}.$$

Show that this bracket satisfies the Jacobi identity.

5.2. **The Goldman bracket.**

Definition 5.11. *Let $\alpha_1, \dots, \alpha_n$ be a set of free loops on a surface S. We say that these loops are in generic position if:*

- *the map*

$$\iota := \alpha_1 \sqcup \cdots \sqcup \alpha_n : (S^1)^{\sqcup n} \to S$$

is a (smooth) immersion

- *there are finitely many intersections and those are transverse double points, i.e. for any intersection point p, $\iota^{-1}(p)$ contains exactly two points, say x, y, and $d\iota(x)$ and $d\iota(y)$ (which are vector tangent to the loops) are linearly independent. We call ι a generic immersion.*

Let $\mathrm{Imm}_0((S^1)^{\sqcup n}, S)$ be the space of generic immersion. This is a dense subset of $\mathrm{C}^\infty((S^1)^{\sqcup n}, S)$ equipped with the Fréchet topology. In particular any multi-loop is homotopic to a generic immersion.

Definition 5.12. *Let α, β be two loops in generic position, let $p \in \alpha \cap \beta$ and let x and y be the (necessarily unique) preimage of p under α and β respectively. Define $\epsilon(\alpha, \beta, p)$ to be $+1$ if $(d\alpha(x), d\beta(y))$ matches the orientation of S and -1 otherwise.*

Definition 5.13 (The Goldman bracket)**.** *Let α, β be in generic position, and define an element of $\mathcal{L}(S)$ by*

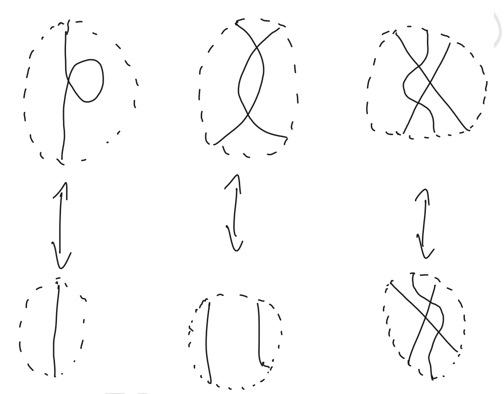
$$[\alpha, \beta] := \sum_{p \in \alpha \cap \beta} \epsilon(\alpha, \beta, p) \overline{\alpha_p \beta_p}$$

where α_p and β_p are the loops based at p associated with α, β and $^-$ means homotopy class.

Theorem 5.14 (Goldman)**.** *This bracket is well-defined (i.e. depends only on the homotopy classes of α and β) and induces a Lie bracket on $\mathcal{L}(S)$, and thus a Poisson bracket on $\mathcal{G}(S)$ via the KKS construction (Example 5.9).*

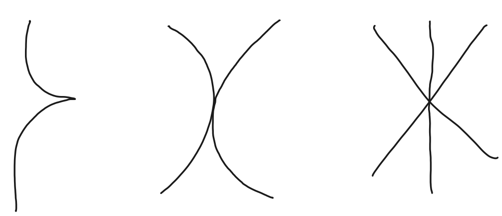
The rest of this section is devoted to the proof of this theorem which is taken pretty much verbatim from [Gol86].

Lemma 5.15. *Let $\iota, \iota' \in \mathrm{Imm}_0((S^1)^{\sqcup n}, S)$ and assume that ι and ι' are homotopic in $\mathrm{C}^\infty((S^1)^{\sqcup n}, S)$. Then there exists a finite sequence $\iota_1 = \iota, \ldots, \iota_k = \iota'$ such that ι_i and ι_{i+1} are related by one of the following moves (with all possible orientations):*

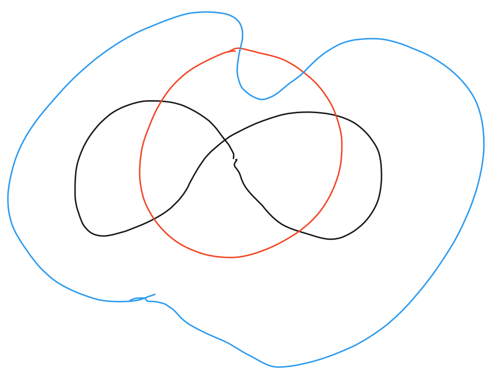


where the pictures mean that we apply the homotopy that is shown inside the dotted disc, and the part outside of this disc is left unchanged.

Proof. Let $\mathrm{Imm}_1((S^1)^{\sqcup n}, S)$ be the space of immersions which are generic except at exactly one point where they look like one of the following picture:



It is clear that such situations cannot be avoided. For example, on the following picture, if one want to move the intersection point of the black loop outside of the red one, at some point there will be a triple point. Of course there are some homotopies for which there will be a quadruple point, e.g. if the intersection point of the black loop goes through the intersection point of the red and blue loop. But clearly this can be avoided.

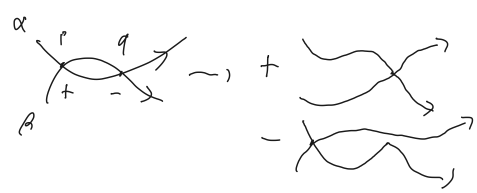


This is actually a general fact: it is a standard result in differential topology (see e.g. [Hir12]) that $\mathrm{Imm}_1((S^1)^{\sqcup n}, S)$ has codimension 1 in $\mathrm{Imm}((S^1)^{\sqcup n}, S)$. Let $\mathfrak{C} \subset \mathrm{C}^\infty((S^1)^{\sqcup n}, S)$ be the space of maps f for which there exists a unique $x \in (S^1)^{\sqcup n}$ such that f is a generic immersion except at x, and such that $df(x) = 0$ but $d^2 f(x) \neq 0$. Then

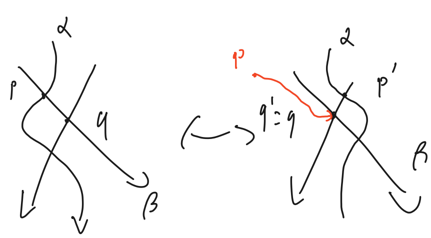
$$\mathrm{C}^\infty{}_1((S^1)^{\sqcup n}, S) := \mathfrak{C} \cup \mathrm{Imm}_1((S^1)^{\sqcup n}, S)$$

has codimension 1 in $\mathrm{C}^\infty((S^1)^{\sqcup n}, S)$. Then, one can choose an homotopy h_t from ι to ι' in such a way hat $h_t \in \mathrm{Imm}_0((S^1)^{\sqcup n}, S)$ except for at most finitely many values of t, in which case it belongs to $\mathrm{C}^\infty{}_1((S^1)^{\sqcup n}, S)$. Thus, around each critical value of t, h_t looks like one of the picture above. $\square$

Since generic homotopies do not change the configuration of intersection points or the homotopy class of the loops based at those points obtained from α, β, it is enough to show that the Goldman bracket is invariant under those moves. It is clearly invariant under the first move since it does not affect the intersections of α and β. The second move might create two intersection points, say p and q (in the case where the two "strands" belong to distinct loops). Let α', β' be the loops obtained by applying this move, then $\alpha'_p \beta'_p$ and $\alpha'_q \beta'_q$ are homotopic, and $\epsilon(\alpha', \beta', p) = -\epsilon(\alpha', \beta', p)$. The following picture shows this situation for one possible orientation:



Finally, the third move can change α, β to α', β' and replace a pair p, q of intersection into a pair p', q' as on the picture (there are other configurations which are dealt with similarly):



Choose a path from p to p' and from q to q' inside the disc. Conjugating by those path takes $\alpha_p, \beta_p, \alpha_q, \beta_q$ to $\alpha'_{p'}, \beta'_{p'}, \alpha'_{q'}, \beta'_{q'}$ respectively. In particular, $\overline{\alpha_p \beta_p} = \overline{\alpha'_{p'} \beta'_{p'}}$ and $\overline{\alpha_q \beta_q} = \overline{\alpha'_{q'} \beta'_{q'}}$. Furthermore, $\epsilon(\alpha, \beta, p) = \epsilon(\alpha', \beta', p')$ and $\epsilon(\alpha, \beta, q) = \epsilon(\alpha', \beta', q')$. Therefore, the bracket is well-defined.

It remains to show that the bracket is antisymmetric and satisfies the Jacobi identity. Note that $\overline{\alpha_p \beta_p} = \overline{\beta_p \alpha_p}$, while exchanging α and β changes the sign of ϵ, hence the bracket is antisymmetric.

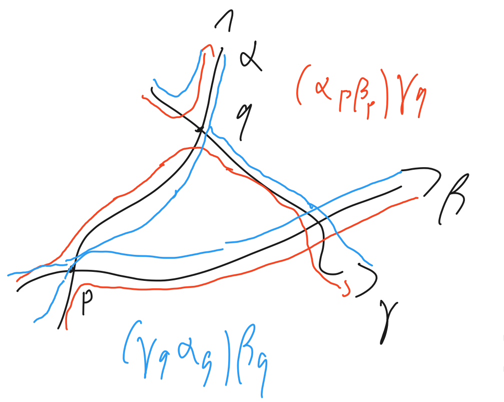
Finally, let α, β, γ be three loops in generic position. The bracket $[[\alpha, \beta], \gamma]$ is a sum over pairs (p, q) where p is an intersection of α and β, and q is an intersection between γ and either α or β. The corresponding term is

$$\epsilon(\alpha, \beta, p)\epsilon(\alpha_p\beta_p, \gamma, q)(\alpha_p\beta_p)_q\gamma_q.$$

Suppose $q \in \alpha$. Then, similarly, in $[[\gamma, \alpha], \beta]$ there is

$$\epsilon(\gamma, \alpha, q)\epsilon(\gamma_q\alpha_q, \beta, p)(\gamma_q\alpha_q)_p\beta_p.$$

One checks that those cancel out: this leads to the same loop on S with opposite signs.



Exercise 5.16. Compute the bracket on a few example. Try to find three loops on some surface whose brackets are pairwise nonzero, and check the Jacobi identity in that case.

Exercise 5.17. Let S be a surface with $\partial S \neq \emptyset$ and let α be a loop around one of the boundary components. Show that for any other loop β, $[\alpha, \beta] = 0$.

6. The Poisson structure on character varieties

6.1. **Kernel of the evaluation map.** We've seen in Proposition 4.9 that there is a canonical evaluation map

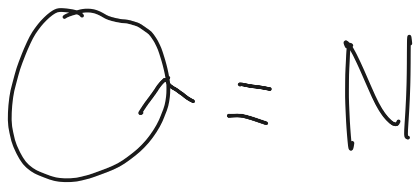
$$\mathrm{ev} : \mathcal{G}(S) \longrightarrow \mathcal{O}(\mathrm{Ch}(S))$$

We claim that this map is surjective, and that its kernel has a nice graphical description. Roughly speaking, among all the traces on the group algebra of the fundamental group of S, we want to characterize those coming from characters of N-dimensional representations. For the sake of clarity we will work with representations on an N-dimensional vector space V over $\mathbb{C}$, but to be perfectly consistent we'd need either to show that $\mathcal{O}(\mathrm{GL}_N(\mathbb{C}))$ is reduced (i.e. does not contain nilpotent) which is true but not completely trivial, or work with $\mathrm{GL}_N(R)$ and $V \otimes_{\mathbb{C}} R$ for arbitrary $\mathbb{C}$-algebras R.

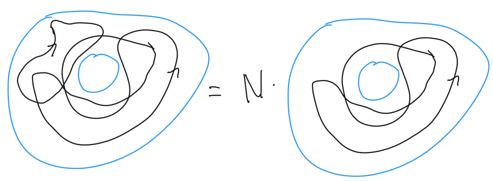
Therefore, roughly speaking, we want to find some abstract way to state that a vector space is of dimension N. We'll actually need two of those. The first one is easy: if $V = R^N$, then

$$\mathrm{Tr}(\mathrm{id}_V) = \dim(V) = N.$$

In other words, for any surface S a trivial loop (i.e. a free loop representing the conjugacy class of $1 \in \pi_1(S, x)$) is always mapped to the constant function $N \cdot 1 \in \mathcal{O}(\mathrm{Ch}(S))$. We'll represent this relation by the following picture:



which we call a 'skein relation". Note this loop is homotopic to a loop with the opposite orientation, so we do not need to add another relation. Of course what we want is the ideal generated by this relation, so really this picture means "if a multi-loop contains a trivial loop, I can delete that loop and multiply what is left by N", e.g.:



The other characterization we need essentially says "in an N-dimensional vector space (or rank N free R-module) any family of $N+1$ vectors satisfies a linear relation". To state this abstractly we need some notations.

First, observe that there is a natural action of the symmetric group S_k on $V^{\otimes k}$ given by

$$\sigma \cdot v_1 \otimes \ldots \otimes v_k := v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(k)}.$$

Definition 6.1. *Let $v \in V^{\otimes k}$. We say that v is antisymmetric if $\forall \sigma \in S_k$,*

$$\sigma \cdot v = \epsilon(\sigma) v$$

where ϵ is the signature. Denote by $\wedge^k V \subset V^{\otimes k}$ the sub-space of antisymmetric elements. Denote by

$$\mathrm{Alt}_k(v) = \frac{1}{k!} \sum_{\sigma \in S_k} \epsilon(\sigma) \sigma \cdot v$$

the natural projection $V^{\otimes k} \to \wedge^k V$

Proposition 6.2. $\wedge^{N+1} V = (0)$.

Proof. Let e_i be a basis of V, then a basis of V^k is given by

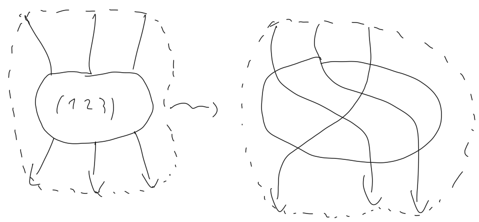
$$e_{i_1} \otimes \ldots \otimes e_{i_k}$$

for all $i_1, \ldots, i_k$. If $k \geq N+1$, any such pure tensor e contains at least two times the same basis element, then antisymmetry implies that

$$\mathrm{Alt}_k(e) = 0$$

in that case. □

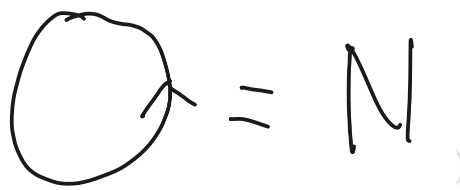
In order to state the main result of this section, we need another notation which is best explained on a example. Suppose we have an oriented graph immersed on S, where every vertex is labelled by some $\sigma \in S_k$ and has exactly k "incoming" followed by k "outgoing" edges in cyclic order. Then we produce a multi-loop on S by replacing each vertex by a diagram representing the associated permutation as follows:

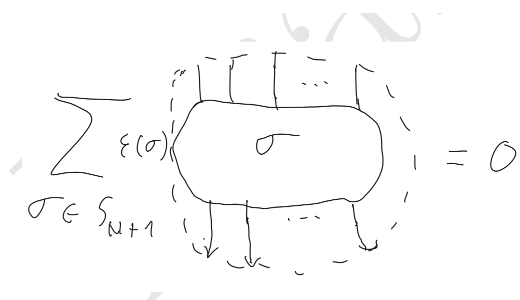


Theorem 6.3 (Sikora [Sik01])**.** *The evaluation morphism*

$$\mathcal{G}(S) \longrightarrow \mathcal{O}(\mathrm{Ch}(S))$$

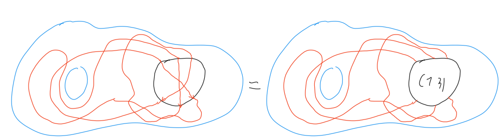
is surjective, and its kernel is generated by the following skein relations:





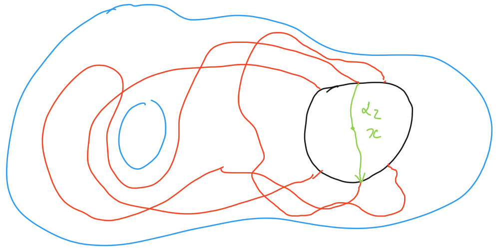
In the second picture the multiloops at hand are assumed to be identical (up to homotopy of course) outside the disc that is shown.

The idea is the following: suppose one choose a disc on S such that each strand which cross the boundary of the disc does so transversally. Up to homotopy it is always possible to move the strands in such a way that each strand enter "at the top" of the disc and leave "at the bottom", in the same order and in such a way that the disc contains a diagram representing a certain permutation σ:



Note that that in the theorem we do *not* require that the strands enter and leave the disc in the same order, but this can always been arranged.

Now, choosing a basepoint inside the disc, any arc which start and ends at the boundary of the disc, can be turned in a unique way into a loop based at that point, let's say $\alpha_1, \ldots, \alpha_k$.



Now, choosing a representation ρ of the fundamental group of S based at p, each α_i induces a linear map

$$\rho(\alpha_i) : V \longrightarrow V.$$

By definition, the function on the character variety induced by this multi-loop maps the class of ρ to

$$\mathrm{Tr}(\sigma \circ (\rho(\alpha_1) \otimes \ldots \otimes \rho(\alpha_k))).$$

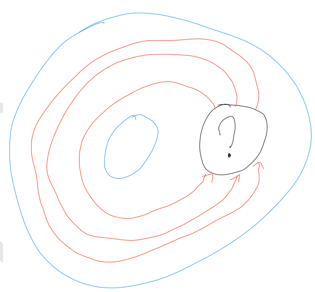
Note that the permutation will in general mix the various copies of V. For example, since traces are multiplicative on tensor product, if f, g are endomorphisms of V,

$$\mathrm{Tr}(f \otimes g) = \mathrm{Tr}(f)\,\mathrm{Tr}(g).$$

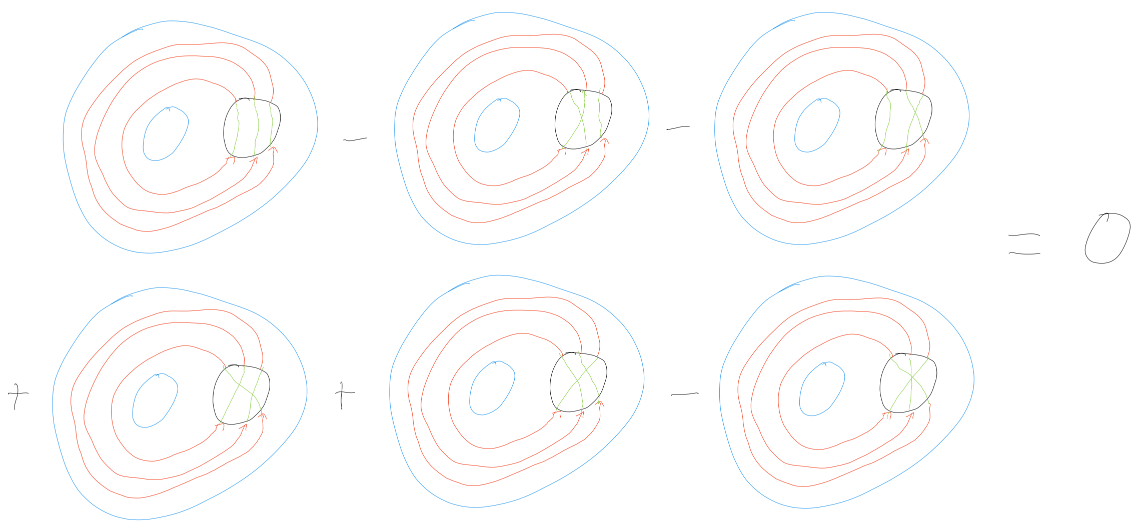
On the other hand,

$$\mathrm{Tr}((1\ 2) \circ (f \otimes g)) = \mathrm{Tr}(f \circ g).$$

Example 6.4. To see an example of the sort of identities produced by the skein relations, consider the following picture:



Then the skein relation say that, in the character variety for GL_2, the following relation holds:



With our choice of orientation, each red arc represents the generator $1 \in \pi_1(\mathbb{A}) \cong \mathbb{Z}$, and any representation maps this generator to some matrix A. Hence this relation becomes:

$$\mathrm{Tr}(A)^3 - \mathrm{Tr}(A^2)\,\mathrm{Tr}(A) - \mathrm{Tr}(A)\,\mathrm{Tr}(A^2) + \mathrm{Tr}(A^3) + \mathrm{Tr}(A^3) - \mathrm{Tr}(A)\,\mathrm{Tr}(A^2) = 0$$

which leads to

$$\mathrm{Tr}(A)^3 - 3\,\mathrm{Tr}(A^2)\,\mathrm{Tr}(A) + 2\,\mathrm{Tr}(A^3) = 0. \tag{1}$$

In other words, this gives a way to write $\mathrm{Tr}(A^3)$ as a polynomial in $\mathrm{Tr}(A)$ and $\mathrm{Tr}(A^2)$. On the other hand, the Cayley–Hamilton theorem states in that case that

$$A^2 - \mathrm{Tr}(A)A + \det(A)\,\mathrm{id} = 0.$$

Taking the trace again, it follows that

$$\det(A) = \frac{1}{2}\left(\mathrm{Tr}(A)^2 - \mathrm{Tr}(A^2)\right).$$

Thinking as det and the Tr_k as symmetric polynomial, this is a particular case of the so-called Newton identities [Wik21]. Plugging this in the characteristic polynomial we obtain this way the fundamental trace identity

$$A^2 - \mathrm{Tr}(A)A + (\frac{1}{2}\left(\mathrm{Tr}(A)^2 - \mathrm{Tr}(A^2)\right))\,\mathrm{id} = 0.$$

Multiplying by $2A$ and taking the trace we get Eq. (1).

Since $G = \mathrm{GL}(V)$ obviously acts on V, it acts on $V^{\otimes k}$ diagonally, i.e.

$$g \cdot (v_1 \otimes \ldots \otimes v_k) = (g \cdot v_1) \otimes \ldots \otimes (g \cdot v_k).$$

Hence also acts on $\mathrm{End}(V^{\otimes k})$ by

$$(g \cdot f)(v) := g \cdot (f(g^{-1} \cdot v)).$$

Observe that the space of invariants for this action is nothing but the space of morphisms of $\mathrm{GL}(V)$-representations from $V^{\otimes k}$ to itself. On the other hand, there is a natural right action of S_k on $V^{\otimes k}$ which clearly commutes with the action of G. In other words, there is a natural algebra map

$$\mathbb{C}[S_k] \longrightarrow \mathrm{End}_{\mathrm{GL}(V)}(V^{\otimes k}).$$

The proof of Theorem 6.3 relies on the following fundamental theorem of representation theory:

Theorem 6.5 (Schur–Weyl duality, [EGH+11, Wey16]). *This map is surjective, and its kernel is 0 if $k \leq N$, and if $k > N$ it is generated as an ideal by*

$$\sum_{\sigma \in S_{N+1} \subset S_k} \epsilon(\sigma)\sigma.$$

For any sequence $i_1, \ldots i_k$, $i_j \in 1 \ldots m$, define

$$\mathrm{Tr}_{i_1 \ldots i_k}(A_1, \ldots, A_m) := \mathrm{Tr}(A_{i_1} \ldots A_{i_k}).$$

Corollary 6.6. *The algebra $\mathcal{O}(\mathrm{End}(V)^{\times m})^G$ is generated by the $\mathrm{Tr}_{i_1 \ldots i_k}$ with relation: for all $k \geq N+1$ and every sequence $i_1, \ldots, i_k$,*

$$\sum_{\sigma \in S_{N+1} \subset S_k} \epsilon(\sigma)\,\mathrm{Tr}(\sigma \circ (A_{i_1} \otimes \ldots \otimes A_{i_k})) = 0.$$

Remark 6.7. In fact, both theorems are equivalent, so we could just as well have admitted this one without proof. However, from the perspective of representation theory, Schur-Weyl duality seems somehow more fundamental (this is subjective of course, though it's probably more well-known), and I find it more intuitive. Also, it is the one that has a clear graphical interpretation similar to what we did above, since we can draw permutations.

In the proof, we will use the following trick: let W be a vector space of dimension n, and choose a basis e_i so that $\mathcal{O}(W) \cong \mathbb{C}[x_1, \dots, x_n]$ with $x_i(e_j) = \delta_{ij}$. Let $x_i^{(j)}$, $j = 1 \dots d$ be a new set of variable and define the symmetrization of a monomial of degree d by

$$\mathrm{Sym}(x_{i_1} \dots x_{i_d}) = \frac{1}{d!} \sum x_{i_1}^{(\sigma(1))} \dots x_{i_k}^{(\sigma(k))}.$$

The right hand side is multilinear, and Sym extends to a linear map from the space of homogeneous polynomial of degree d to the space of d-multilinear forms on W. For example,

$$\mathrm{Sym}(x^3 + 2xy^2) = x^{(1)}x^{(2)}x^{(3)} + \frac{2}{3}x^{(1)}y^{(2)}y^{(3)} + \frac{2}{3}x^{(3)}y^{(1)}y^{(2)} + \frac{2}{3}x^{(2)}y^{(3)}y^{(1)}$$

where x, y is a basis of $W^* = (\mathbb{C}^2)^*$ and $x^{(i)}$ means we apply this linear form on the ith component. Conversely, given a d-multilinear form μ one gets an homogeneous polynomial $\mathrm{Res}(\mu)$ by replacing each $x_i^{(j)}$ by x_i, and by construction

$$\mathrm{Res}(\mathrm{Sym}(P)) = P.$$

Remark 6.8. A more abstract way to think about this construction is as follow: let $T(W)$ be the *tensor algebra* on W. Formally this is the free non-commutative algebra on W. As a vector space this is

$$T(W) = \bigoplus_{i \geq 0} W^{\otimes i}$$

and the multiplication is given by $a \cdot b = a \otimes b$. By definition, $O(W) = S(W^*)$ is thus the quotient of $T(W^*)$ by the ideal generated by $(u \otimes v - v \otimes u)$ for $u, v \in T(W^*)$. This quotient has a section (as a vector space) which maps the commutative product $u_1 u_2 \dots u_d$ of elements $u_i \in W^*$ in the degree d part of $\mathcal{O}(W)$) to the symmetrization

$$\frac{1}{d!} \sum_{\sigma \in S_d} u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \dots \otimes u_{\sigma(d)}.$$

By construction this lands in $(W^*)^{\otimes d} \subset T(W^*)$ which, using the universal property of the tensor product, is nothing but the space of d-multilinear forms on W.

Now if G acts on W, the induced action on $\mathcal{O}(W)$ preserves each space of homogeneous polynomial and Sym and Res are maps of G-representations. In particular, P is G-invariant if and only if $\mathrm{Sym}(P)$ is.

Let $d_1, \dots, d_m$ be integers and set $d = \sum d_i$. Then, there is more generally a restitution map Res from $(W^{\otimes d})^*$, the space of multi-linear maps on W^d to the space of polynomial maps on W^m which are multi-homogeneous of degree $(d_1, \dots, d_m)$ (i.e. homogeneous of degree d_i with respect to the ith copy of W). This map is given by replacing the first d_1 variables by the corresponding variables on the first copy of W, the next d_2 variables by the corresponding variables on the second copy of W etc. This map is again G-equivariant, and has an equivariant section given by applying Sym successively for each copy of W.

proof (of the corollary). We apply this to $W = \mathrm{End}(V)$. Observe that the trace induces a non-degenerate pairing

$$\mathrm{End}(V^{\otimes d}) \times \mathrm{End}(V^{\otimes d}) \longrightarrow \mathbb{C}$$

given by

$$(A, B) \longmapsto \mathrm{Tr}(AB).$$

This in turn induces an isomorphism

$$\mathrm{End}(V^{\otimes d}) \longrightarrow \mathrm{End}(V^{\otimes d})^*$$

given by

$$A \longmapsto (X \mapsto \mathrm{Tr}(AX)).$$

This map is clearly a morphism of G representation because $\forall g \in G$

$$\mathrm{Tr}(gAg^{-1}X) = \mathrm{Tr}(Ag^{-1}Xg).$$

Thus, G-invariant homogeneous polynomial functions of degree $(d_1, \dots, d_m)$ on $\mathrm{End}(V)^m$ are identified with $\mathrm{End}_G(V^{\otimes d})$, and the result then follows from Schur–Weyl duality. For example, the map

$$\begin{aligned} \mathrm{End}(V)^3 &\longrightarrow \mathbb{C} \\ (A, B, C) &\longmapsto \mathrm{Tr}(ABC) \end{aligned}$$

is multi-linear, and applying the various versions of Res for $m = 2$ one gets the polynomial functions on W^2 given by $\mathrm{Tr}(X^3), \mathrm{Tr}(XY^2), \mathrm{Tr}(X^2Y)$ and $\mathrm{Tr}(Y^3)$ respectively, which are of multi-degree $(3, 0), (1, 2), (2, 1), (0, 3)$. ☐

We now prove Theorem 6.3, first for punctured surfaces. In that case we have

$$\mathcal{O}(\mathrm{Ch}(S)) \cong \mathcal{O}(\mathrm{GL}(V)^m)^{GL(V)} \cong \mathcal{O}(\mathrm{End}(V)^m)^{\mathrm{GL}(V)}[\frac{1}{\det_i}, i = 1 \dots m]$$

where $m = 2g + n - 1$ and where g is the genus of S and n the number of boundary components.

It follows from the previous result that this is generated by traces of products of matrices *and their inverses* (which accounts for the $\det_i^{-1}$'s) modulo the skein relation.

If S is a closed surface, let S° be S with a disc removed. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{G}(S^\circ) & \longrightarrow & \mathcal{G}(S) \\ \big\downarrow{\scriptstyle \mathrm{ev}} & & \big\downarrow{\scriptstyle \mathrm{ev}} \\ \mathcal{O}(\mathrm{Ch}(S^\circ)) & \longrightarrow & \mathcal{O}(\mathrm{Ch}(S)) \end{array}$$

obtained by seeing a loop in S° as a loop in S. The two horizontal maps and the leftmost vertical one are surjective, hence so is the rightmost one, and it shows its kernel is also generated by the same skein relation.

6.2. **Poisson structure on** $\mathrm{Ch}(S)$**.**

Definition 6.9. *Let $(\mathcal{O}, \{\ ,\ \})$ be a Poisson algebra. An ideal $P \subset \mathcal{O}$ is called a Poisson ideal if*

$$\{\mathcal{O}, P\} \subset P.$$

Proposition 6.10. *Let $P \subset \mathcal{O}$ be a Poisson ideal. Then the Poisson structure on $\mathcal{O}$ canonically descends to a Poisson structure on $\mathcal{O}/P$, and the map $\mathcal{O} \to \mathcal{O}/P$ is Poisson.*

Proof. clear. ☐

Theorem 6.11. *Let S be a surface. The kernel of the evaluation map $\mathrm{ev} : \mathcal{G}(S) \to \mathcal{O}(\mathrm{Ch}(S))$ is a Poisson ideal.*

Proof. This is almost tautological, and shows the usefulness of using skein/local relations. Let P be the kernel of ev, $p \in P$ and $x \in \mathcal{G}(S)$. We need to show that

$$\{p, x\} \in P.$$

Note that P is in fact *linearly* generated by all the skein relations (i.e. the product of a skein relation by an arbitrary element of $\mathcal{G}(S)$ is again a skein relation). Note also that it is enough to check this in the case x is a loop, although this actually do not simplify the proof much. Indeed, if $y \in \mathcal{G}(S)$ then the derivation property shows

$$\{p, xy\} = y\{p, x\} + x\{p, y\}$$

so if $\{p, x\} \in P$, since P is in particular an ideal then also $y\{p, x\} \in P$, hence so does $\{p, xy\}$. Now if x is a loop and p is a linear combination of multiloops which are identical except maybe inside a fixed disc on S (or even a finite number of discs), then clearly we can arrange so that all intersections between x and the components of p lie outside those discs. In particular, this shows that if $p \in P$, then indeed $\{p, x\} \in P$. ☐

7. Cutting and gluing

In this section, we'll see a combinatorial way to describe character varieties and their Poisson structures by roughly speaking cutting surfaces in elementary pieces and then gluing those pieces back together along part of their boundary. This is, in a somewhat loose sense, an example of what mathematicians call a topological field theory[4].

7.1. **Fundamental groupoid and representations.** The character varieties themselves are actually *not* compatible with gluing for the following fairly obvious reason: in general, if S is obtained by gluing say S' and S'' (we'll make a precise definition of what it means later on) then there are loops on S that are not images of loops in either S' or S'' (Figure 7.1).

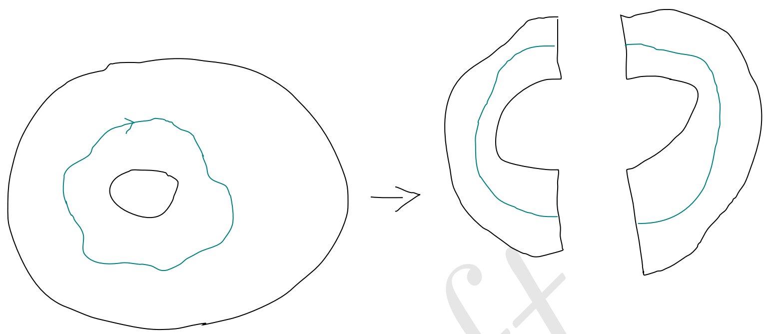


Figure 1. This loop in an annulus is not the image of a loop in either half-annuli.

As the picture suggests, we actually need to allow not just loops but paths between points on the boundary components. This is the main reason for the following definitions. Let S be a non-necessarily connected surface every component of which has a boundary, and X a finite subset of ∂S which contains at least one point on each connected component of S. We will often call X a *marking* of X and the pair (S, X) a *marked surface.*

Definition 7.1. *The fundamental groupoid $\Pi_1(S, X)$ is the category whose set of objects is X, and whose set of morphisms from x to y is given by homotopy classes of paths in S from x to y.*

Definition 7.2. *A representation of $\Pi_1(S, X)$ on a vector space V is a functor from $\Pi_1(S, X)$ to the category of vector space, which maps every object $x \in X$ to V. In other words, this is a morphism of groupoids from $\Pi_1(S, X)$ to $\mathrm{GL}(V)$ seen as a groupoid with a single object.*

Unpacking the definition, this just means we pick a copy V_x of our fixed vector space for every x in X, and for every homotopy class of a path $x \to y$ we choose an isomorphism $V_x \to V_y$ in such a way that composition of paths goes to compositions of maps the in the obvious way.

Remark 7.3. Both definitions are of course much less general than they could/should be, X could be an arbitrary subset of S and a representation could be an arbitrary functor to vector spaces but those will be the versions we'll need.

Let $G_x = \mathrm{GL}(V_x)$, $x \in X$, and let ρ be a representation of $\Pi_1(S, x)$. Then any $M = (M_x)_{x\in X} \in \prod G_x$ gives a new representation ρ^M such that for any path γ from x to y

$$\rho^M(\gamma) = M_y \rho(\gamma) M_x^{-1}$$

Of course, at this point, we could choose a presentation by generators and relations of $\Pi_1(S, X)$, use it to put an algebraic variety structure on the set of representations of this groupoid and then recover the character variety of S by taking a categorical quotient as we did before. But as it turns out, the whole point of this section is to introduce some combinatorial data that will actually makes these choices for us so we don't need to.

[4]To turn this into an *actual* example of a topological field theory, one needs to work with character *stacks* rather than character varieties, but then what it even means to have a Poisson structure is much more complicated.

7.2. **Skeletons and fusion.** The basic combinatorial object we will need is the following.

Definition 7.4. *A skeleton on (S, X) is a graph Γ embedded on S, whose vertex set is X, and such that S is a deformation retract of Γ.*

The reader is invited to look at the very well-written paper [LBŠ15] for many examples with much better pictures and more.

Proposition 7.5. *Every marked surface (S, X) has a skeleton. Every choice of an orientation on the edges of a skeleton determines a set of generators for $\Pi_1(S, X)$.*

Proof. We've seen that any connected surface with non-empty boundary is diffeomorphic to a disjoint union of "standard" surfaces as on Figure 7.2.

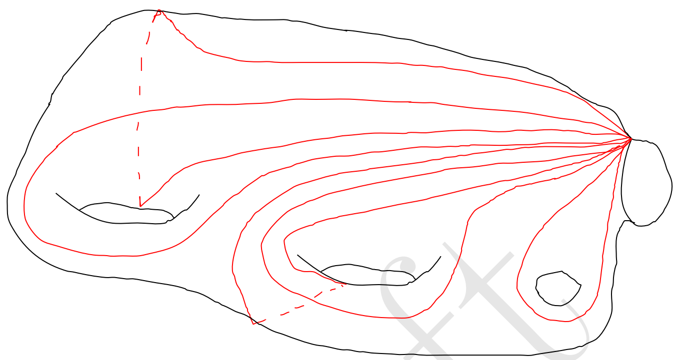


FIGURE 2. A one vertex skeleton of $S_{2,2}$.

Now choosing a basepoint on the "outer" boundary, and the specific free generators of the fundamental group that we used gives a skeleton with one vertex. We can then pull everything back through the chosen diffeomorphism.

The second point follows by definition of skeletons, since the surface at hand deformation retracts on the underlying graph. □

Let us emphasize again that none of this is canonical, and that there are many choices of a skeleton.

Observe that an orientation on S canonically induces an orientation on each component of ∂S. Indeed, let C be one such component, then there is a unique orientation on C such that at any chosen point $x \in C$, the basis of the tangent space at x made of a vector tangent at x to C pointing in the same direction as the orientation, and a vector tangent at x to S, orthogonal to the previous one and pointing towards the inside of S, is oriented the same way as S.

Let P be a compact oriented 1-dimensional manifolds (i.e. a disjoint union of intervals and circles) and ∂_1, ∂_2 two smooth embeddings of P in ∂C.

Definition 7.6. *The gluing of S along $(P, \partial_1, \partial_2)$ is the manifold obtained by modding out S by the relation $\{\partial_1(x) = \partial_2(x), x \in P\}$ and smoothing the resulting topological space the obvious way.*

Remark 7.7. Smoothing is required when gluing over an interval, see below.

Definition 7.8. *Let (S, X, Γ) be a marked surface with a skeleton. Let $x, y \in X$ be distinct, then the* fusion *of S at (x, y) is the marked surface with a skeleton obtained by gluing S along two small marked intervals on the boundary respectively ending at x and y (so in particular those vertices are identified after this process).*

Proposition 7.9. *Every marked surface with a skeleton can be obtained by iterative fusion of a collection of copies of a disc with a two vertices skeleton as in Figure 7.2.*

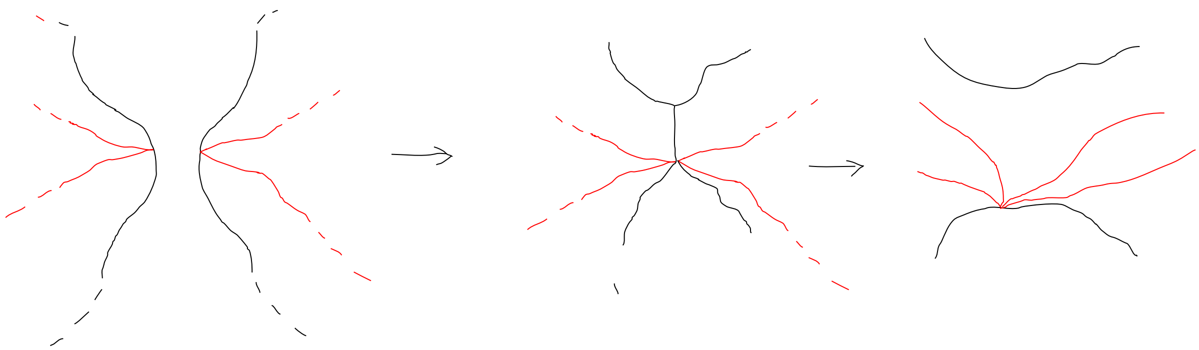


FIGURE 3. Fusion followed by smoothing.

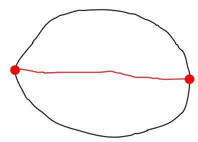


FIGURE 4. The basic piece

Proof. Choose a small piece of surface around each edge, as on the left-hand-side of Figure 7.2, in such a way that all of those pieces are disjoint (this is possible since the edges of the graph don't intersect). Make each piece bigger and bigger until they cover the surface without intersecting as on the right hand side of the same figure (this is possible because the surface deformation-retracts on the graph). Each of these piece is thus diffeomorphic to a copy of the standard disc of Figure 7.2, and it's clear that the surface we started with is obtained by fusion of all of those pieces, by cutting along the lines between two regions. $\square$

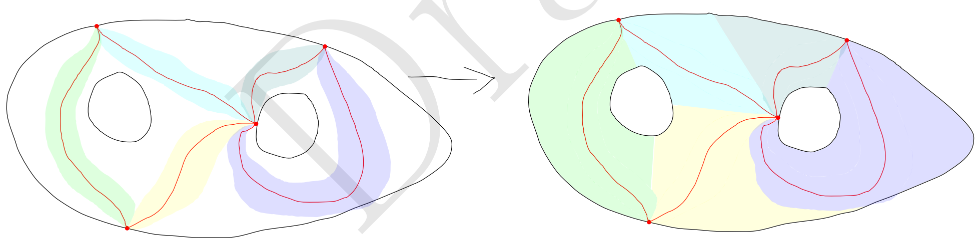


FIGURE 5. Cutting a surface with a skeleton into standard discs.

Proposition 7.10. *Every choice of an orientation on the edges of a skeleton on S with set of vertices X gives an identification (of algebraic variety)*

$$\mathrm{Rep}(\Pi_1(S, X)) \cong G^{|E|}$$

where E is the set of edges of edges.

Proof. By construction, once chosen an orientation on the edges, any representation ρ of $\Pi_1(S, X)$ gives an element of G for every edge e from x to y given by

$$\rho(e) : V = V_x \to V_y = V.$$

On the other hand, any choice of elements of G $(g_e)_{e \in E}$ determines a representation of $\Pi_1(S, X)$ (i.e. the edges clearly form a set of free generators for the fundamental groupoid). $\square$

The main idea behind all of this is that if a skeletal surface S is obtained by fusion from S' then the corresponding representation varieties are literally the same, which of course would not be true had we worked with the fundamental group (because the number of edges stays the same), although they carry actions of different groups (since the number of vertices decreases when fusing).

8. Quasi-Poisson structures

8.1. The Lie algebra $\mathfrak{gl}_n$. Recall that we denoted by $\mathfrak{gl}_n$ the Lie algebra of matrices equipped with the bracket given by the commutator. There is a general way of associating a Lie algebra to an affine algebraic group, and clearly $\mathfrak{gl}_n$ should be the Lie algebra of GL_n. We refer to [Per] for the general theory and just work out the particular case we need.

Recall that for a commutative algebra $\mathcal{O}$, $\mathrm{Der}(\mathcal{O})$ is the space of derivation.

Proposition 8.1. *$\mathrm{Der}(\mathcal{O})$ is a Lie algebra, with bracket given by the commutator of endomorphisms.*

Proof. Clear. □

Definition 8.2. *An action of $\mathfrak{gl}_n$ on a commutative algebra $\mathcal{O}$ is a Lie algebra map*

$$\mathfrak{gl}_n \to \mathrm{Der}(\mathcal{O}).$$

Example 8.3. Let $(\mathcal{O}, \{,\})$ be a Poisson algebra and $\rho : \mathfrak{gl}_n \to \mathcal{O}$ a morphism of Lie algebra. Then $\mathfrak{gl}_n$ acts on $\mathcal{O}$ by

$$a \cdot x := \{\rho(a), x\}.$$

Proposition 8.4. *Let X be a variety with algebra $\mathcal{O}$ and suppose that it carries an algebraic action of $\mathrm{GL}_n(\mathbb{C})$. Then this induces an action of $\mathfrak{gl}_n$ on $\mathcal{O}$ which maps a to the derivation d_a defined by*

$$d_a(f) = \frac{d}{dt}\left((1+ta)\cdot f\right)\Big|_{t=0}$$

There are two actions of G on itself inducing actions on $\mathcal{O}(G)$, given by multiplication on the left and on the right respectively. Hence, if $A \in G$ and $f \in \mathcal{O}(G)$ we have operators

$$(A^L \cdot f)(x) := f(A^{-1}x) \qquad\qquad (A^R \cdot f)(x) := f(xA).$$

Note that these actions actually commute. By the previous proposition, we get two actions of $\mathfrak{gl}_n$ on $\mathcal{O}(G)$. For $a \in \mathfrak{gl}_n$ we'll denote by a^L and a^R the corresponding derivations. They are uniquely characterized by the fact that for $\lambda \in \mathfrak{gl}_n^*$, seen as an element of $\mathcal{O}(G)$,

$$(a^L \cdot \lambda)(x) := \lambda(-ax) \qquad\qquad (a^R \cdot \lambda)(x) := \lambda(xa).$$

Using again the trick of viewing the generators of $\mathcal{O}(G)$ as a matrix $X = (x_{ij})$ of variables, then we see that $a^R \cdot x_{ij}$ is simply the (i,j) coefficient of the matrix Xa (and likewise for a^L).

For every $a \in \mathfrak{gl}_n, B \in G$, the left action of a commutes with the right action of B and vice versa. In fact, one can show that $\mathfrak{gl}_n$ is isomorphic to the sub-Lie algebra of $\mathrm{Der}(\mathcal{O}(G))$ which are invariant under the action by left (resp. right) multiplication of G on itself (this, in fact, is the proper definition of the Lie algebra of an affine algebraic group).

We also have the adjoint action, the action of G on itself by conjugation. The induced action of $\mathfrak{gl}_n$ is given by $a \mapsto a^L + a^R$.

8.2. The Schouten bracket. Let $\mathfrak{g}$ be a Lie algebra, and let

$$\bigwedge \mathfrak{g} = \bigoplus_{p\geq 0} \wedge^p \mathfrak{g}.$$

For $u \in \wedge^p\mathfrak{g}$, we set $|u| = p$ and we say it is of degree p. The wedge product

$$\wedge^p\mathfrak{g} \times \wedge^q\mathfrak{g} \longrightarrow \wedge^{p+q}\mathfrak{g}$$

induces an algebra structure on $\bigwedge \mathfrak{g}$. This product is graded-commutative in the sense that for elements u, v of degree p, q respectively,

$$u \wedge v = (-1)^{[p||q|} v \wedge u.$$

Proposition 8.5. *There is a unique extension, which we call the Schouten bracket, to $\bigwedge \mathfrak{g}$ of the bracket of $\mathfrak{g}$ such that for u, v, w of degree p, q, r respectively*

- $[u,v] = -(-1)^{(p-1)(q-1)}[v,u]$
- $[u, v\wedge w] = [u,v]\wedge w + (-1)^{(p-1)q} v\wedge [u,w]$
- $(-1)^{(p-1)(r-1)}[u,[v,w]] + (-1)^{(q-1)(p-1)}[v,[w,u]] + (-1)^{(r-1)(q-1)}[w,[u,v]] = 0.$

where we declare that if u is of degree 0 (i.e. $u \in \mathbb{C}$) then $[u,v]=0$. An explicit formula for the bracket is:

$$[u_1\dots u_k, v_1\dots v_l] = \sum_{i,j}(-1)^{i+j}[u_i,v_j]u_1\dots\hat{u_i}\dots u_k v_1\dots\hat{v_j}\dots v_l$$

where for readability we suppress the $\wedge$ symbol and where $\hat{}$ means that this element is removed.

Proof. Clearly these two properties can be used to inductively define the bracket of any two elements, and the general formula then follows from an easy induction. Using this formula, the last relation follows from the Jacobi identity for the bracket on $\mathfrak{g}$. $\square$

Remark 8.6. These properties can be thought of as graded versions of antisymmetry, the derivation property and the Jacobi identity of a Poisson bracket. This makes $\bigwedge \mathfrak{g}$ into a so-called Gerstenhaber algebra, nowadays often also named "2-Poisson algebra" of "P_2-algebra" (see [LGPV12, Section 3.3.3]). This should be thought of as a version of the KKS construction but with a grading shift. Note also that the bracket is of degree -1, i.e. it restricts to maps

$$\wedge^p\mathfrak{g}\times\wedge^q\mathfrak{g} \longrightarrow \wedge^{p+q-1}\mathfrak{g}$$

where by convention $\wedge^{-1}\mathfrak{g} = \{0\}$. This is where the "2" comes from: there is a general notion of a n-Poisson algebra, which is a graded-commutative algebra with something like a Poisson bracket of degree $1-n$.

Likewise, let $\mathcal{O}$ be a commutative algebra and let $\mathfrak{X}^p(\mathcal{O})$ be the space of linear maps

$$\wedge^p\mathcal{O}\to\mathcal{O}$$

which are derivation in each variables. In particular,

$$\mathfrak{X}^1(\mathcal{O}) = \mathrm{Der}(\mathcal{O})$$

and

$$\mathfrak{X}^0(\mathcal{O}) = \mathcal{O}.$$

Also, Poisson brackets on $\mathcal{O}$ are in particular elements of $\mathfrak{X}^2(\mathcal{O})$. Let $\mathfrak{X}(\mathcal{O}) = \bigoplus_{p\geq 0}\mathfrak{X}^p(\mathcal{O})$ and define grading the same way as before.

A (p,q) shuffle is a permutation $\sigma\in S_{p+q}$ such that

$$\sigma(1)<\sigma(2)<\dots<\sigma(p) \qquad \text{and } \sigma(p)<\sigma(p+1)<\dots<\sigma(p+q).$$

Let $S_{p,q}$ be the set of (p,q)-shuffles.

Definition 8.7. *The shuffle product on $\mathfrak{X}(\mathcal{O})$ is given by*

$$(u\wedge v)(x_1,\dots,x_{p+q}) := \sum_{\sigma\in S_{p,q}} u(x_{\sigma(1)},\dots,x_{\sigma(p)})u(x_{\sigma(p)},\dots,x_{\sigma(p+q)}).$$

Example 8.8. if $u,v\in\mathrm{Der}(\mathcal{O})$, then

$$(u\wedge v)(x,y) = u(x)v(y) - u(y)v(x).$$

Proposition 8.9. *There is a unique extension of the Lie bracket on* $\mathrm{Der}(\mathcal{O})$ *to a degree -1 bracket on* $\mathfrak{X}(\mathcal{O})$ *satisfying and characterized by the same properties as in Proposition 8.5, together with*

- $\forall x,y\in\mathcal{O}, [x,y]=0$
- $\forall x\in\mathcal{O}, d\in\mathrm{Der}(\mathcal{O}), [d,x]=d(x).$

given by the explicit formula for $u\in\mathfrak{X}^p(\mathcal{O}), v\in\mathfrak{X}^q(\mathcal{O})$:

$$[u,v](x_1,\dots,x_{p+q-1}) = \sum_{\sigma\in S_{q,p-1}}\epsilon(\sigma)u\left(v(x_{\sigma(1)},\dots,x_{\sigma(q)}),x_{\sigma(q+1)},\dots,x_{\sigma(q+p-1)}\right)$$
$$-(-1)^{(p-1)(q-1)}\sum_{\sigma\in S_{p,q-1}}\epsilon(\sigma)v\left(u(x_{\sigma(1)},\dots,x_{\sigma(p)}),x_{\sigma(p+1)},\dots,x_{\sigma(q+p-1)}\right)$$

Remark 8.10. Proposition 8.5 can be obtained from this one, by identifying $\bigwedge \mathfrak{g}$ with the G-invariant part (for the action by right multiplication, say) of $\mathfrak{X}(\mathcal{O}(G))$. More generally, if $\mathfrak{g}$ acts on $\mathcal{O}$ then we have a map

$$\bigwedge \mathfrak{g} \longrightarrow \mathfrak{X}(\mathcal{O})$$

compatible with the product and the bracket on each side, where $u = u_1 \dots u_p$ acts by

$$u(x_1, \dots, x_p) = \sum_{\sigma \in S_p} \epsilon(\sigma) u_1(x_{\sigma(1)}) \dots u_p(x_{\sigma(p)}).$$

One of the main reason for defining the Schouten bracket is the following direct consequence of the explicit formula above:

Proposition 8.11. *Let $\mu \in \mathfrak{X}^2(\mathcal{O})$. Then $\forall x, y, z \in \mathcal{O}$*

$$[\mu, \mu](x, y, z) = -2 \left(\mu(x, \mu(y, z)) + \mu(z, \mu(x, y)) + \mu(y, \mu(z, x)) \right).$$

In particular, μ is a Poisson bracket on $\mathcal{O}$ iff $[\mu, \mu] = 0$.

8.3. **Quasi-Poisson algebras.** Recall that the trace induces a non-degenerate, G-invariant symmetric pairing on the vector space $\mathfrak{gl}_n$, hence an isomorphism of G-representations $\mathfrak{gl}_n \cong \mathfrak{gl}_n^*$. Let $(e_{ij})_{1 \leq i, j \leq n}$ be the elementary matrices, then $(e_{ji})_{1 \leq i, j \leq n}$ is the dual basis with respect to this pairing.

Definition 8.12. *Let t be the canonical element associated with this pairing, i.e. the image of the identity under the isomorphism*

$$\mathrm{End}(\mathfrak{gl}_n) \cong \mathfrak{gl}_n \otimes \mathfrak{gl}_n^* \cong \mathfrak{gl}_n \otimes \mathfrak{gl}_n.$$

Explicitly,

$$t = \sum_{i,j} e_{ij} \otimes e_{ji}.$$

Definition 8.13. *Let*

$$\phi = \frac{1}{4} [t \otimes \mathrm{id}, \mathrm{id} \otimes t]$$

where the bracket here means the commutator of endomorphisms in $\mathfrak{gl}_n^{\otimes 3} = \mathrm{End}(V^{\otimes 3})$. Explicitly,

$$\phi = \frac{1}{4} \sum_{i,j,k} \left(-e_{ij} \otimes e_{jk} \otimes e_{ki} + e_{jk} \otimes e_{ij} \otimes e_{ki} \right).$$

Proposition 8.14. *The element ϕ is antisymmetric and G-invariant.*

Proof. This follows at once from the antisymmetry of the bracket, and the G-invariance and symmetry of t. □

Let $\mathcal{O}$ be a commutative algebra with an action of G, inducing an action of $\mathfrak{gl}_n$.

Definition 8.15. *A G-quasi-Poisson structure on $\mathcal{O}$ is the data of a bracket $\{\ ,\ \}$ which:*

- *is bilinear, antisymmetric, and G-equivariant, i.e.*

$$\forall g \in G, g \cdot \{x, y\} = \{g \cdot x, g \cdot y\}$$

- *is a derivation in each variable*
- *satisfies the following weakening of the Jacobi identity:*

$$\{x, \{y, z\}\} + \{y, \{z, x\}\} + \{z, \{x, y\}\} = \phi(x, y, z).$$

where $\phi(x, y, z)$ is the map $\mathcal{O}^{\otimes 3} \to \mathcal{O}$ induced by the action of $\mathfrak{g}$ as in Remark 8.10.

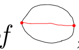
Remark 8.16. The bracket being G-equivariant implies that for $u \in \mathfrak{gl}_n$,

$$u \cdot \{x, y\} = \{u \cdot x, y\} + \{x, u \cdot y\}.$$

More generally, if $\mathcal{O}$ carries an action of G^n for some integer n, then this induces an action of the Lie algebra $\mathfrak{gl}_n^{\oplus n}$ on $\mathcal{O}$, and we can define the notion of G^n-quasi-Poisson algebra.

Exercise 8.17. Show that if x, y, z are G-invariant, then $\phi(x, y, z) = 0$. Show that the sub-algebra $\mathcal{O}^G$ is thus a Poisson algebra.

Let us give our first, seemingly trivial but in fact really important, example of a quasi-Poisson variety.

Proposition 8.18. *The representation variety of* , *equipped with the zero bracket, is a $G \times G$-quasi Poisson variety.*

Proof. Recall that this variety is just G, with $G \times G$-actions given by the actions by left and right multiplication respectively, so the induced action of $\mathfrak{gl}_n \oplus \mathfrak{gl}_n$ is given by

$$(u, v) \mapsto u^L + v^R.$$

Recall that these actions are determined by their actions on the generators, hence for λ, μ, ν linear forms on $\mathfrak{gl}_n$ seen as functions on G, and for $A \in A$, we have

$$(\phi^L(\lambda, \mu, \nu))(A) = (\lambda \otimes \mu \otimes \nu)(\phi \cdot (A \otimes A \otimes A)) \qquad (\phi^R(\lambda, \mu, \nu))(A) = -(\lambda \otimes \mu \otimes \nu)((A \otimes A \otimes A) \cdot \phi)$$

where $\cdot$ is just the usual multiplication of matrices. But recall that Φ is actually G invariant, where G acts on $\mathfrak{gl}_n$, hence on $\mathfrak{gl}_n^{\otimes 3}$, by conjugation, so that ϕ actually commutes with $A \otimes A \otimes A$. Therefore, we get that $\phi^L = -\phi^R$, so that $\phi^L + \phi^R = 0$. $\square$

8.4. **Fusion.** Recall the element $t = \sum_{i,j} e_{ij} \otimes e_{ji} \in \mathfrak{gl}_n^{\otimes 2}$. We define an element

$$\psi = \frac{1}{2}(t^{1,2} - t^{2,1}) = \frac{1}{2}\sum_{i,j}(e_{ij}, 0) \wedge (0, e_{ji}) \in \wedge^2(\mathfrak{gl}_n \oplus \mathfrak{gl}_n)$$

where the superscript ^{1} means the corresponding component goes to the first copy of $\mathfrak{gl}_n$ and ^{2} to the second copy. Just like t is the canonical element of the symmetric pairing on $\mathfrak{gl}_n$ induced by the trace, ψ is the canonical element for the *skew*-symmetric pairing on $\mathfrak{gl}_n \oplus \mathfrak{gl}_n$ given by:

$$((u_1, u_2), (v_1, v_2)) \longmapsto \frac{1}{2}\left(\mathrm{Tr}(u_1 v_2) - \mathrm{Tr}(u_2 v_1)\right).$$

We will also denote by the superscript ^{diag} for the diagonal map $x \mapsto (x, x) = x^1 + x^2 \in \mathfrak{gl}_n \oplus \mathfrak{gl}_n$ and we will use the same superscripts for their extensions to $\wedge \mathfrak{gl}_n$. The following is key:

Proposition 8.19. *We have*

$$-\frac{1}{2}[\psi, \psi] = \phi^{diag} - \phi^1 - \phi^2.$$

Proof. The formula for the Schouten bracket gives (suppressing sums for clarity)

$$\frac{1}{2}[\psi, \psi] = \frac{1}{8}[e_{ij}, e_{kl}]^1 \wedge e^1_{ji} \wedge e^2_{lk} + [e_{ij}, e_{kl}]^2 \wedge e^1_{ji} \wedge e^1_{lk}$$

By definition, for $u, v \in \mathfrak{gl}_n$, $[u^1, v^2] = 0$, i.e. elements in different copies of $\mathfrak{gl}_n$ don't interact. Recall that

$$\phi = \frac{1}{4}[t \otimes 1, 1 \otimes t] := \frac{1}{4}(e_{ij} \otimes [e_{ji}, e_{kl}] \otimes e_{lk}).$$

Since it is antisymmetric, we have

$$\phi = \frac{1}{24} e_{ij} \wedge [e_{ji}, e_{kl}] \wedge e_{lk} = -\frac{1}{24}[e_{ij}, e_{kl}] \wedge e_{ji} \wedge e_{lk}$$

(we also used that t is symmetric to switch e_{ij} and e_{ji}). Using again that element in different copies commute, and the fact that t is symmetric, we get

$$\begin{aligned}
\phi^{diag} &= -\frac{1}{24}[e^1_{ij} + e^2_{ij}, e^1_{kl} + e^2_{kl}] \wedge (e^1_{ji} + e^2_{ji}) \wedge (e^1_{lk} + e^2_{kl}) \\
&= -\frac{1}{24}\left([e_{ij}, e_{kl}]^1 \wedge e^1_{ji} \wedge e^1_{lk} + [e_{ij}, e_{kl}]^2 \wedge e^2_{ji} \wedge e^2_{lk}\right) \\
&\quad - \frac{1}{8}\left([e_{ij}, e_{kl}]^1 \wedge e^1_{ji} \wedge e^2_{lk} + [e_{ij}, e_{kl}]^2 \wedge e^1_{ji} \wedge e^1_{lk}\right)
\end{aligned}$$

where the factor 3 comes out from gathering terms, since antisymmetric tensors are invariant under 3-cyclic permutations. Hence,

$$\phi^{diag} = \phi^1 + \phi^2 - \frac{1}{2}[\psi, \psi]$$

as required. $\square$

This allows us to introduce a certain operation which, basically says that fusion of marked surfaces can be done in a way compatible with quasi-Poisson structures.

Theorem 8.20 (Alekseev–Kosmann-Schwarzbach–Malkin-Meinrenken [AKSM02])**.** *Let $\mathcal{O}$ be a G^2-quasi-Poisson algebra with bracket μ. Then, equipped with the new bracket*

$$\{x, y\}^{fus} = \mu(x, y) - \psi(x, y)$$

and the diagonal G action, is a G-quasi-Poisson algebra. This operation is called fusion.

Proof. This new bracket is clearly bilinear, antisymmetric and G-invariant, because μ and ψ are. By definition, and using the relation between the Schouten bracket and Jacobi,

$$-\frac{1}{2}[\mu, \mu] = \phi^1 + \phi^2$$

where the superscript indicate the actions of the respective copies of G. Hence,

$$\begin{aligned}
[\mu - \psi, \mu - \psi] &= [\mu, \mu] - 2[\mu, \psi] + [\psi, \psi] \\
&= [\mu, \mu] + [\psi, \psi] \text{ since } \mu \text{ is } G \times G \text{ invariant by assumption} \\
&= -2(\phi^1 + \phi^2) + 2(\phi^1 + \phi^2) - 2\phi^{diag} \text{ by Proposition 8.19.}
\end{aligned}$$

Therefore,

$$-\frac{1}{2}[\mu - \psi, \mu - \psi] = \phi^{diag}$$

as required. $\square$

Example 8.21. As an example, we can consider fusion of the zero quasi-Poisson structure on the representation variety of a disc with two marked points. We thus get a G-quasi Poisson structure on G with G-action given by conjugation, whose bracket is given by the action of

$$-\frac{1}{2}\sum e^L_{ij} \wedge e^R_{ji} = \frac{1}{2}\sum e^R_{ij} \wedge e^L_{ji}.$$

Note that this is nothing but the representation variety of a disk with one hole and one marked point on the boundary ! We can compute this bracket explicitly (see e.g. [MT14, Appendix B]). Recall that $\mathcal{O}(G)$ is generated by the linear forms $x_{i,j}$ defined by $x_{ij}(e_{k,l}) = \delta_{i,k}\delta_{j,l}$, so that the bracket is determined by its value on those. As explained at the send of section 8.1, the $\mathfrak{gl}_n$ actions by derivation on linear forms is easy to compute, hence:

$$e^R_{i,j} \cdot x_{k,l} = -\delta_{i,k} x_{j,l} \qquad\qquad e^L_{i,j} \cdot x_{k,l} = \delta_{j,l} x_{k,i}.$$

It follows that

$$\{x_{i,j}, x_{k,l}\} = \frac{1}{2}\sum_m \left(-\delta_{j,k} x_{i,m} x_{m,l} + \delta_{i,l} x_{k,m} x_{m,j}\right).$$

Finally, we get the following combinatorial construction of a quasi-Poisson structure on representations varieties:

Theorem 8.22 (Alekseev–Kosmann-Schwarzbach–Meinrenken [AKSM02], Li-Bland–Severa [LBŠ15])**.** *The representation variety of any marked surface (S, V) carries a canonical G^V-quasi-Poisson structure, induced by the action of an element $\mu_{S,V} \in \bigwedge \mathfrak{gl}_n^{\oplus|V|}$ and uniquely characterized by the following properties:*

- *For the disc with two marked points, the bracket is zero.*
- *If (S, V) is the disjoint union of (S', V') and (S'', V''), then*

$$\mu_{S,V} = \mu_{S',V'} + \mu_{S'',V''}.$$

- *If (S,V) is obtained from (S',V') by fusion of two vertices v, v', then $\mu_{S,V}$ is obtained from $\mu_{S',V'}$ by fusion along the inclusion $G^v \times G^{v'} \subset G^V$.*

This, in particular, induces a Poisson structure on the character variety by taking G^d-invariants, which coincides with the one coming from the Goldman bracket.

We refer to [LBŠ15, Nie13] and in particular to [MT14, Appendix B] for a detailed proof. Basically, note that there was a in fact an a priori more general way to construct function on the character variety of a surface S (with non-empty boundary say): take $f \in \mathcal{O}(G)^G$ any invariant function, γ a loop on S and define $f_\gamma \in \mathcal{O}(\mathrm{Ch}(S))$ by

$$f_\gamma(\rho) = f(\rho(\gamma)).$$

What we showed is that we in fact do not need to do that, it's enough to restrict to $f = \mathrm{Tr}$, but if we do we can somehow reduce the number of loops we need. For example, if α is the generator of the fundamental group of an annulus, we can take $\gamma = \alpha^2$ and define as we did the function $\rho \mapsto \mathrm{Tr}(\rho(\gamma))$. But of course since ρ is a morphism, this equals $\mathrm{Tr}(\rho(\alpha)^2)$. Hence, we could equally well think of this as the function f_α, with $f \in \mathcal{O}(G)^G$ being the function $X \mapsto \mathrm{Tr}(X^2)$. In fact, Goldman theorem really is about this more general situation, which also works for groups others than GL_N.

Now passing to the representation variety with respect to a set of basepoints on the boundary on S, we can do the same trick using non-necessarily invariant functions (of course it will then depends on the base point). That is, if $f \in \mathcal{O}(G)$ is any function, and γ a path on S between two basepoints then we have a function f_γ mapping now an actual (as opposed to an equivalence class of) representation ρ to $f(\rho(\gamma))$. Of course we recover functions on the character variety by restricting to those functions which are in fact invariant.

What we gain by doing this, is that it is now enough to restrict to generators of the fundamental groupoid to get all functions on the representations variety, and on the other side we can also restrict to generators of $\mathcal{O}(G)$. This is really just a way of saying again that a choice of a skeleton Γ on S with set of edges E and set of vertices V gives an isomorphism

$$\mathcal{O}(G^E) \longrightarrow \mathcal{O}(\mathrm{Rep}(S,V))$$

which, in the notations we just introduced, is determined by

$$\forall e \in E, x_{ij}^{(e)} \longmapsto (x_{ij})_e := (\rho \mapsto x_{ij}(\rho(e)))\,.$$

Long story short, the references above defined a quasi-Poisson version of Goldman's formula, which makes sense for arbitrary functions on the representation variety, and which of course reduces to Goldman's formula in case the functions are invariant. It is then enough to compare the value of this bracket, to the one obtained by iterative fusion, on those generators.

9. Quantization and knot theory

9.1. **Quantizations of Poisson algebras.** Let $\hbar$ be a variable (the "Planck constant"). Let $\mathbf{K} = \mathbb{C}[[\hbar]]$ be the ring of formal power series in $\hbar$. Its elements are of the form

$$\sum_{n\geq 0} \hbar^n z_n, z_n \in \mathbb{C}$$

and the multiplication is defined by

$$(\sum_{n\geq 0} \hbar^n z_n) \cdot (\sum_{n\geq 0} \hbar^n z'_n) = \sum_{n\geq 0} \hbar^n (\sum_{p+q=n} z_p z'_q).$$

Let W be a vector space (possibly infinite dimensional). We define

$$W[[\hbar]] := \left\{ \sum_{n\geq 0} \hbar^n v_n, v_n \in V \right\}.$$

This is naturally a $\mathbf{K}$-module.

Definition 9.1. *Let* $(\mathcal{O},\cdot)$ *be a commutative algebra. A* star product *on* $\mathcal{O}$ *is a linear map*

$$\star:\mathcal{O}\otimes\mathcal{O}\longrightarrow\mathcal{O}[[\hbar]$$

such that:

(1) *The composition*

$$\mathcal{O}[[\hbar]]\otimes_{\mathbf{K}}\mathcal{O}[[\hbar]]\longrightarrow(\mathcal{O}\otimes\mathcal{O})[[\hbar]]\longrightarrow\mathcal{O}[[\hbar]],$$

where the first map is the canonical inclusion and the second map is the unique extension of $\star$ *into a map of* $\mathbf{K}$*-module, induces an associative multiplication on* $\mathcal{O}[[\hbar]]$.

(2) *For* $a,b\in\mathcal{O}$

$$a\star b=a\cdot b+O(\hbar).$$

In other words, the canonical projection induces an algebra isomorphism

$$(\mathcal{O}[[\hbar]],\star)/(\hbar)\cong(\mathcal{O},\cdot).$$

Proposition 9.2. *Let* $\star$ *be a star product on* $\mathcal{O}$ *and set*

$$\forall a,b\in\mathcal{O},\{a,b\}:=\frac{a\star b-b\star a}{\hbar}\mod\hbar.$$

Then, $(\mathcal{O},\{,\})$ *is a Poisson algebra.*

This gives a "purely algebraic" motivation for the notion of Poisson algebra: those arises when studying how to deform commutative algebras into non-commutative one.

Definition 9.3. *In the setting of the previous proposition, we call* $\mathcal{O}$ *the classical limit of* $(\mathcal{O},\star)$*,* $(\mathcal{O},\{,\})$ *its quasi-classical limit, and we call* $(\mathcal{O},\star)$ *a quantization of* $(\mathcal{O},\{,\})$*.*

Example 9.4. Again this notion is originally inspired by physics. Recall the Poisson algebra $(\mathbb{C}[x,p])$ with bracket defined by $\{x,p\}=1$. Let $\hat{x},\hat{p}$ be the endomorphisms of $\mathbb{C}[x][[h]]$ defined by "multiplication by x" and $-\hbar\frac{\partial}{\partial x}$ respectively. Let $D_\hbar(\mathbb{C})$ be the Weyl algebra, i.e. the subalgebra generated by $\hat{x},\hat{p}$ and $\hbar$ with multiplication given by the composition of endomorphisms. We claim this is a quantization of this Poisson algebra.

For $g\in\mathbb{C}[x]$ we have

$$(\hat{x}\hat{p})\cdot g=-\hbar x\frac{\partial g}{\partial x}$$

while

$$(\hat{p}\hat{x})\cdot g=-\hbar\frac{\partial(xg)}{\partial x}=-\hbar x\frac{\partial g}{\partial x}-\hbar g.$$

More generally, if $g\in\mathbb{C}[x]$ is regarded as an element of $D_\hbar(\mathbb{C})$ (we replace x by $\hat{x}$) then

$$\hat{p}^n g=g\hat{p}^n-\hbar^n\frac{\partial^{n-1}}{\partial^{n-1}x}g.$$

Therefore, every element of this algebra can be written uniquely as

$$\sum_{n\geq0}\hbar^nP_n$$

where

$$P_n=\sum_{i,j=0}^{k_n,l_n}\alpha_{ij}^{(n)}\hat{x}^i\hat{p}^j,\alpha_{ij}^{(n)}\in\mathbb{C}$$

In other words, there is a $\mathbf{K}$-module isomorphism

$$D_\hbar(\mathbb{C})\cong\mathbb{C}[\hat{x},\hat{p}][[\hbar]]\cong\mathbb{C}[x,p][[\hbar]].$$

We can pull back the multiplication of $D_\hbar(\mathbb{C})$ via this isomorphism to define a star product $\star$ on $\mathbb{C}[x,p]$, and the commutation relation above implies that

$$\frac{x\star p-p\star x}{\hbar}=1+O(\hbar)=\{x,p\}+O(\hbar)$$

as required.

Roughly speaking, the function x and p corresponds to the operations “measure the position” and “measure the momentum” respectively. In quantum mechanics, those are promoted to linear operators (as opposed to functions) on some space, and setting $\hbar$ to be the actual Planck constant ($\simeq 6.62 \times 10^{-34}$), the relation

$$[\hat{x}, \hat{p}] = \hbar$$

becomes Heisenberg uncertainty principle. Like wise, the Hamiltonian is promoted to a “quantum Hamiltonian” $\hat{H} \in D_\hbar(\mathbb{C})$ and the evolution of some operator Ψ_t is given by

$$\frac{\partial}{\partial t}\Psi_t = [\hat{H}, \Psi_t].$$

For example, a quantum Harmonic oscillator is governed by

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{x}^2$$

in which case the above equation becomes the Schrödinger equation.

Remark 9.5. This particular example makes it sound like it is “easy” to find a quantization of a given Poisson algebra, but this is very non true in general. A famous theorem due to Kontsevich states that every Poisson structure on a polynomial algebra in finitely many variable can be quantized, but this is a notoriously hard result.

9.2. **The algebra of links in S.** Fix a surface S and let $I = [0, 1]$ b an interval.

Definition 9.6. *A link in S is a smooth embedding of a (possibly empty) union $(S^1)^{\sqcup n}$ of n circles into $S \times I$. If $n = 1$ this is often called a knot.*

Recall that a smooth embedding is an immersion (i.e. a smooth map whose derivative is nowhere zero) which is an homeomorphism onto its image. In particular, as the name clearly suggest, an embedding is injective. So roughly speaking, we “thicken” the surface by crossing it with an interval, so that we get a 3-dimensional space, and we embed a bunch of circle in that space in such a way that the strands never intersect.

Definition 9.7. *Two links are said to be equivalent (often one says they are the “same” link) if there is an isotopy between them, i.e. a path in the space of smooth embedding.*

In other words, two links are equivalent if one can smoothly deform one into the other without ever allowing strand to intersect (in particular, they are not allowed to cross).

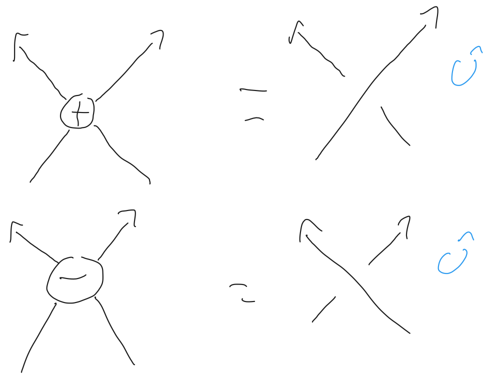
The study of knots and links is highly non-trivial (and an active part of modern mathematics) already in the case where S is a disc. The most basic problem is to find efficient ways to tell whether two links are equivalent. This is often done by constructing invariants, that is of functions on the space of links which take the same value on equivalent links. It turns out that most of the invariants that are studied nowadays have deep and surprising connections to *à priori* unrelated areas of mathematics (representation theory and category theory to name a few) and theoretical physics (quantum and conformal field theories).

Definition 9.8. *Let $\mathrm{Lk}(S)$ be the vector space of formal $\mathbb{C}$-linear combinations of equivalence classes of links.*

Proposition 9.9. *The space $\mathrm{Lk}(S)$ has a natural structure of an associative algebra, whith multiplication given by “stacking” two copies of $S \times I$ containing some links one top of each other, and then rescaling.*

Definition 9.10. *A* link diagram *on S is the data of a multi-loop on S in generic position, together with a choice for each crossing of a labelling by $+$ or $-$.*

The label $+$ and $-$ represents positive and negative crossings, which one usually draw like this (using the chosen orientation on the surface):



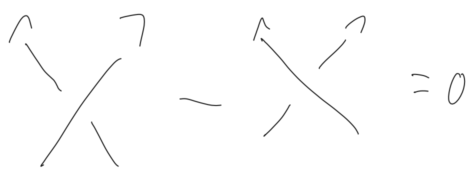
To any multiloop in generic position, one associates a graph on S by turning any intersection point into a vertex. Note that this graph is embedded in S, while the loo is *à priori* only immersed.

Definition 9.11. *A* diagram isotopy *between two diagrams is an isotopy between the underlying embedded graphs which preserves the labels at each vertex.*

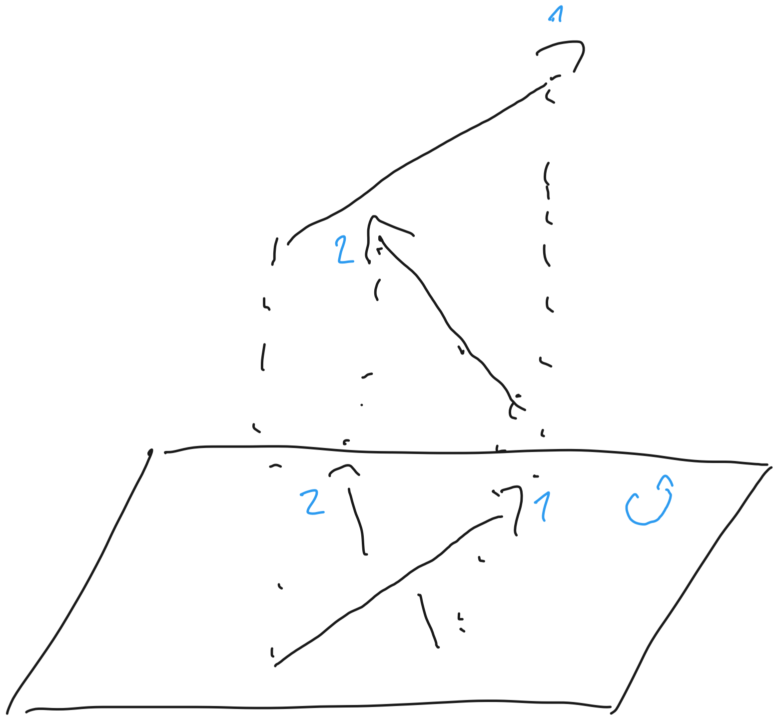
Let π_S be the projection $S \times I \to S$ along I. The projection of a link is a multi-loop on S. Note that π_S is an homotopy equivalence: it induces a bijection between homotopy classes of multiloops on S, and homotopy class of multiloops in $S \times I$ which, in turn, is the same as the set of homotopy (as opposed to isotopy) classes of links on S. This can be rephrased as follows: π_S induces an algebra map

$$\mathrm{Lk}(S) \longrightarrow \mathcal{G}(S)$$

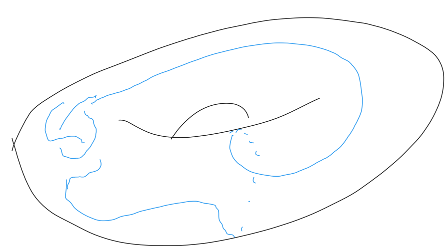
whose kernel is generated by the skein relation



Definition 9.12. *A link L is said to be in generic position, if the multi-loop $\pi_S(L)$ is in generic position. This is turned into a diagram by declaring that the label of an intersection is $+$ is the strand which is first in the chosen orientation (using the right hand rule) is above the second one, and $-$ otherwise.*



Here is an example of a diagram of a knot on a torus:

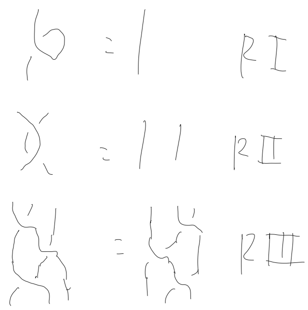


Proposition 9.13. *Let L be a link on S in generic position. Then L is determined, up to isotopy, by its diagram.*

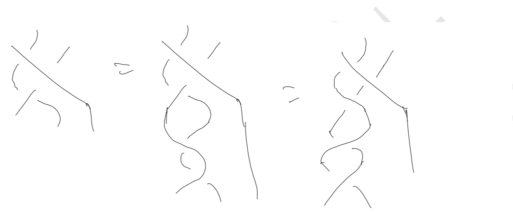
Proof. One can lift the underlying multi-loop into a loop in $S\times]0,1[$ which will have some intersection. Then there is a unique way to "resolve" each intersection point depending on the label on that intersection point. $\square$

The following fundamental theorem turns the question of whether two links are isotopic, into a combinatorial question about diagrams.

Theorem 9.14 (Reidemeister)**.** *Every link in S is isotopic to one in generic position. Two links in S are isotopic if and only if any pair of diagrams representing them are related by the successive applications of diagram isotopies and the following moves (for every choice of orientation of the strands):*



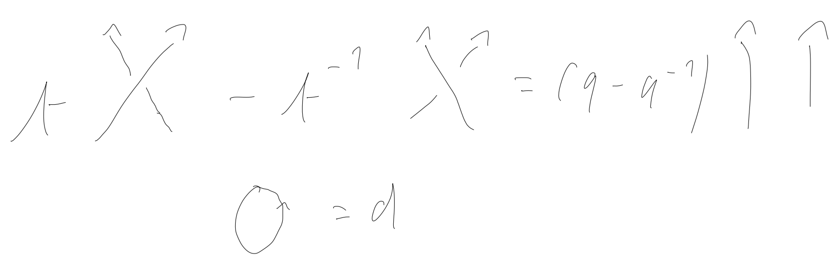
Sketch of proof. This follows from similar arguments as in Theorem 5.14. The space of links in generic position is open and dense in the space of links (e.g. because π_S is an open map, and the space of links in generic position is the preimage of the space of multiloops in generic position). Then one shows that the space of links whose projection is in generic position except at exactly on point where there is either a cusp, a tangent or a triple point is of codimension 1. Hence any isotopy between two links is homotopic to one which intersect this space transversally and only finitely many time. Every isotopy that do not intersect this space clearly induces a diagram isotopy of the underlying diagram. Then any possible singular point is resolved by applying the Reidemeister moves. Note that it seems we need, e.g., other versions of RIII but they can be obtained from the one we have using RII:



□

9.3. **The HOMFLY skein algebra.** We want to produce a quantization of the Goldman bracket using the algebra of links in S. However it is clear this algebra is "much bigger" than the algebra of loops since there are many different links that project onto a given loop. Also, as we observed, the very reason this algebra is non-commutative is precisely because strands are not allowed to cross. So we want to somehow impose some extra relation that will allow us to flip crossing at the cost of some expression involving $\hbar$. We'll start with an algebraic version of this idea, which we'll then expand by plugging some expression depending on $\hbar$.

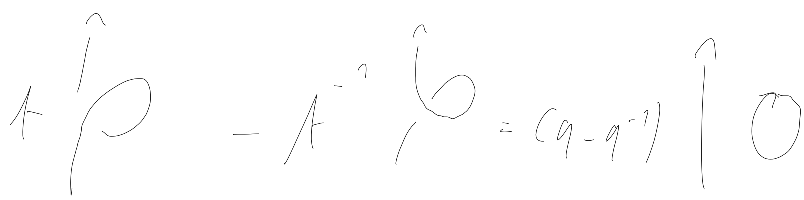
Definition 9.15. *Let R be a commutative algebra, and let $q, t, d \in R$ be such that q, t are invertible. The HOMFLY skein algebra $\mathrm{Sk}_R(S)$ of S is the quotient of the space $\mathrm{Lk}_R(S)$ of formal R-linear combinations of isotopy classes of links in S by the following skein relations:*



Remark 9.16. These relations are given by linear combination of diagrams, so here we are implicitly identifying the space of links with the space of diagrams modulo the Reidemeister relations. We will see later that this is consistent in the sense that if one takes the quotient of the space of R-linear combination of diagrams by the skein relation, then the Reidemeister relations are automatically satisfied. In other words, every relation obtained by applying Reidemeister moves to a skein relation is itself already a skein relation.

Remark 9.17. There various choices and normalisations involved in the definition of the HOMFLY skein algebra. Behind the curtain, just like the Goldman algebra has something to do with the GL_N for all N, this algebra is closely related to something called "quantum GL_N", for which there is a version of Schur–Weyl duality involving the Hecke algebra. The normalisation we chose is consistent with this representation theoretic perspective.

Remark 9.18. The first relation implies in particular the following identity



which, using the second relation and the fact that all relations obtained from the skein relations by applying Reidemeister moves are also satisfied, implies

$$(t - t^{-1}) \uparrow = d(q - q^{-1}) \uparrow$$

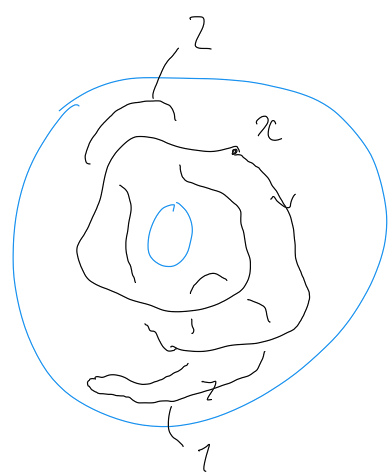
Therefore, we can and will assume that

$$t - t^{-1} = d(q - q^{-1}).$$

In particular, if $(q - q^{-1})$ is invertible, this forces the value of d.

Definition 9.19. *Let D be a link diagram on S. An ordering of D is a choice of an ordering of its components. One says D is based it a basepoint is chosen on each component. An ordered and based diagram is called ascending if when considering components one after the other in the chosen order, starting from the basepoint and moving along the link following the orientation, each crossing is first met by "going under".*

Here is an example of an ascending diagram, where the numbers indicate the order of the components:



Observe that an ascending diagram is necessarily "not knotted" as illustrated by the following not ascending diagram:

In particular, the components of an ascending diagram are unlinked since by definition each component lies above the ones preceding it in the given order.

Theorem 9.20 ([FYH$^+$85, Prz92]). *Let R, q, t, d be as in Definition 9.15 and assume that*

$$t - t^{-1} = d(q - q^{-1}).$$

Let $\hat{\pi}^\circ = \hat{\pi} - \{1\}$. There exists an isomorphism of R-module

$$\iota : \mathrm{Sk}_R(S) \cong R[\hat{\pi}^\circ]$$

such that every multi-loop α is mapped to a certain link $L\alpha$ with $\pi_S(L_\alpha) = \alpha$.

9.4. **Quantization of the Goldman algebra.** We are going to prove a version of this theorem for a simpler version of the skein module, which is enough to quantized the Goldman algebra.

Definition 9.21. *Two links are said to be link-homotopic if there is an homotopy between them whith the property that strands between* distinct components *never intersect. Let $h\,\mathrm{Lk}(S)$ be the space of $\mathbb{C}$-linear combinations of link-homotopy classes of links in S.*

In other words, we forget about the knottedness of each individual component, and remember only how the components are linked between each other.

Definition 9.22. *The homotopy skein module $\mathrm{hSk}_\hbar(S)$ is the quotient of $h\,\mathrm{Lk}(S)[[\hbar]]$ by the skein relation:*

where on the left hand side the two strands belongs to distinct components (otherwise it would be incompatible with the link homotopy relation).

In other words, we take the quotient of $\mathrm{Lk}(S)[[\hbar]]$ by the above skein relations for crossings between different components, and the relation saying that for crossings between strands belonging to the same component, the left hand side of the skein relation is zero.

Let $<$ be a total order on $\hat{\pi}(S)$.

Theorem 9.23 ([Tur91]). *There is a unique isomorphism of $\mathbb{C}[[\hbar]]$-module*

$$H : \mathrm{hSk}_\hbar(S)) \cong S(\hat{\pi}(S))[[\hbar]]$$

characterized by the following property: if D is an ascending diagram for a link L whose component are (in the order induced by the ordering of the components of D) $L_1, L_2, \ldots, L_k$ such that $\pi_S(L_i) \leq \pi_S(L_{i+1})$, then

$$H(L) = \prod_{i=1}^{n} \pi_S(L_i).$$

Proof. Let D_n be the space D_n of $\mathbb{C}$-linear combinations of diagrams of links on S with at most n crossings. We will construct by induction a map H_n from $D_n[[h]]$ whith the following properties:

(1) it is invariant under the skein relations, under the Reidemeister moves involving only diagrams with at most n-crossings and the link-homotopy relations.
(2) if D is a diagram with strictly less than n crossings, then $H_n(D) = H_{n-1}(D)$.
(3) H_n satisfies the property indicated in the theorem.

If D is an arbitrary diagram with 0 crossing, then it means the knots represented by each component commute in $h\,\mathrm{Lk}(S)$ so we can assume they are ordered in a way compatible with the chosen order on $\hat{\pi}(S)$. The resulting diagram is then ascending with respect to any choice of a basepoint on each component, so that the value of H_0 is determined by the third property we're requiring. Property (1) and (2) are vacuous in that case.

Suppose we have constructed H_{n-1}. Then, for any diagram D which can be made ascending for certain choices of basepoints and of a certain order compatible with $<$, we again define $H_n(D)$ using property (3). The only ambiguity in the choice of an order is if two components represents the same element of $\pi_1(S)$, in which case the corresponding component commute so if one swap them in the chosen order, the result is the same. Clearly, the value do not depends on the choice of basepoints either.

Now if D is an arbitrary diagram, choose an order of its components compatible with $<$ and a basepoint on each component. Once this choice is made, there is a unique way to "flip" some of the crossings on D (i.e. change a positive crossing into a negative one of vice versa) to make it ascending. Starting with this ascending diagram, whose value we know, we start applying crossing flips to get to D. At each step, either the crossing is between two pieces of the same component, in which case the value of H_n on the diagram obtained this way is the same as the original diagram, of they belong two different components. In this case, we use the skein relation to define the value of H_n on the newly created diagram. Indeed, in the skein relation

$$D_+ - D_- = \hbar D_0$$

where either D_+ or D_- is the diagram we had before applying the flip, D_0 has strictly less than n crossings so $H_n(D_0)$ is known by induction hypothesis. Therefore, we know the value of H_n on two out of three of the diagrams involved in the skin relation, hence we can use it to define the value of H_n on the third.

This is compatible with the skein relation in the sense that for any sequence of crossings flips between two diagrams D, D', if one define the value of H_n on D' using that on D using the skein relation, one always the same value. Indeed, it is enough to check this by applying the skein relation on two different crossings in the two possible orders. Suppose, e.g., that those two crossings are positive and let $D_{\alpha,\beta}$ be the diagram obtained from D by replacing the first (resp. the second) crossing by α (resp. β) in $\{+,-,0\}$. If we resolve the first crossing first, the skein relation gives

$$\begin{aligned} H_n(D_{++}) &= H_n(D_{-+}) + \hbar H_n(D_{0+}) \\ &= H_n(D_{--}) + \hbar H_n(D_{-0}) + \hbar H_n(D_{--}) + \hbar^2 H_n(D_{00}). \end{aligned}$$

If we start with the second crossing,

$$\begin{aligned} H_n(D_{++}) &= H_n(D_{+-}) + \hbar H_n(D_{+0}) \\ &= H_n(D_{--}) + \hbar H_n(D_{00}) + \hbar H_n(D_{-0}) + \hbar^2 H_n(D_{00}). \end{aligned}$$

This does not depend on the choice of a basepoint on each component since the image of a given component depends only of its projection, not on the underlying knot. Sine we only allow Reidemeister moves with at most n-crossing, the only version of RI we need to check is the one that remove a little loop. Clearly, if the diagram to which we applied this move was ascending, then so is the one obtained this way.

Finally, one checks that if two diagrams are related by either RII or RII, then this process produces the same value for both diagrams.

This shows H is well-defined, and also that in $\mathrm{hSk}_\hbar(S)$ any link can be written in a unique way as a $\mathbb{C}[\hbar]$-linear combination of link homotopy classes of links, each of which is an ordered product $K_1 \dots K_k$ where $\pi_S(K_i) \leq \pi_S(K_{i+1})$ (again, we're using that up to link-homotopy K_i is determined by its projection). $\square$

Hence, we can pull-back the product of $\mathrm{hSk}_\hbar(S)$ to get a star product $\star$ on $S(\hat{\pi}(S))$.

Corollary 9.24. *This is a quantization of the Goldman algebra of S.*

Proof. The previous theorem shows, in particular, that this algebra is generated as an algebra by images of knots, i.e. by loops. Let α, β be loops on S such that $\alpha \leq \beta$. Let K_α, K_β be any knot lifting those loops and let L be the product $K_\alpha K_\beta$ and L' be the product $K_\beta K_\alpha$. Then by definition $\alpha \star \beta = H(L) = \alpha\beta$ while in order to compute $\beta \star \alpha = H(L')$ we need to use the skein relation to move K_β past K_α from the top to the bottom. Choosing a diagram for L', it means we need to flip every crossing between its two component. By definition there is exactly one crossing for each intersection point between α, β. Let D be the diagram obtained at some step of this process, and let p be an intersection point which has not been processed, and assume p corresponds to a positive crossing. Let D' be the diagram obtained by flipping this crossing. The projection of the diagram obtained by removing this crossing as on the right hand side of the skein relation (which is then the diagram of a knot), is homotopic to $\alpha_p\beta_p$. Hence the skein relation implies that

$$H(D) = H(D') + \hbar\alpha_p\beta_p.$$

The case where p is a negative crossing is identical except this produces a minus sign in front of $\hbar\alpha_p\beta_p$. Therefore, we have shown that

$$\alpha \star \beta - \beta \star \alpha = \hbar \sum_{p \in \alpha \cap \beta} \epsilon(\alpha, \beta, p)\alpha_p\beta_p$$

as required. □

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