FACTORIZATION HOMOLOGY OF BRAIDED TENSOR CATEGORIES (THIS VERSION: July 5, 2023)

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Abstract.

Contents

0. Introduction

1. Warm-up: sheaves on character varieties and skein categories

1.1. **Recollection on reductive groups and quotients.** We will always work over $\mathbb{C}$. Let $\mathcal{O}$ be a finitely generated commutative algebra and R an arbitrary commutative algebra. Any choice of a finite presentation

$$\mathcal{O} \simeq \mathbb{C}[x_1, \ldots, x_n]/I$$

induces a bijection between the set

$$Z(R) := \mathrm{Hom}_{\mathrm{alg}}(\mathcal{O}, R)$$

and the set of common zeros in R^n for elements of I

$$\{z \in R^n \mid \forall f \in I, f(z) = 0\}.$$

This is clearly functorial in R, and by Yoneda Lemma the functor $Z(-)$ determines $\mathcal{O}$ up to canonical isomorphism. In particular, such a presentation identifies $Z(\mathbb{C})$ with a set of zeroes of polynomials in $\mathbb{C}^n$. We call Z the variety associated with $\mathcal{O}$, and $\mathcal{O}$ the algebra of function on Z. In other words, for the purpose of these notes, "variety" means "affine scheme of finite type over $\mathbb{C}$". Likewise a morphism of variety is just a morphism (in the opposite direction) between their respective associated algebras.

An affine algebraic group is a variety, G equipped with a multiplication given by a map

$$\Delta : \mathcal{O}(G) \longrightarrow \mathcal{O}(G)^{\otimes 2}$$

and an inverse given by a map

$$S : \mathcal{O}(G) \to \mathcal{O}(G)$$

inducing on $\mathcal{O}(G)$ the structure of an Hopf algebra.

Definition 1.1. *An algebraic representation of G is a right $\mathcal{O}(G)$-comodule. We denote by* $\mathrm{Rep}\, G$ *the category of algebraic representations of G.*

Let us emphasize that we do not impose any finite dimensionality condition so this is a "large" category: the collection of isomorphism classes of objects is not a set. If $V \in \mathrm{Rep}\, G$ then V carries a linear action of G[1] in the usual sense: for $v \in V$ and letting

$$\Delta(v) = \sum_i f_i \otimes v_i$$

be the coaction on v, the action is given by

$$g \cdot v = \sum_i f_i(g) v_i.$$

The converse is not true in general (not all representations of G are algebraic), but this is clearly true for finite-dimensional representations. Their role is emphasized by the following striking property of categories of comodules:

Theorem 1.2 (Fundamental theorem of comodules)**.** *Let C be a coalgebra over a field. Then every C-comodule is the union of its finite-dimensional sub-comodules.*

Corollary 1.3. *Let V be a linear representation of G. Then V is algebraic iff it is locally finite dimensional, i.e. if for every $v \in V$, there exists a finite dimensional subrepresentation of V that contains v.*

Definition 1.4. *An affine algebraic group is called* reductive *if the category* $\mathrm{Rep}\, G$ *is semisimple.*

[1] really a linear action of $G(\mathbb{C})$ but we won't make that distinction

Example 1.5. Let x_{ij}, $1 \leq i, j \leq n$ be indeterminates which we'll often conveniently think of as coefficients of a matrix

$$X = (x_{ij}).$$

We set

$$\mathcal{O}(\mathrm{GL}_n) := \mathbb{C}[X, \det(X)^{-1}].$$

It's easily seen that there is a natural bijection for any commutative algebra R

$$\mathrm{GL}_N(R) \simeq \mathrm{Hom}_{\mathrm{alg}}(\mathcal{O}(\mathrm{GL}_n), R)$$

showing that, although we picked a particular presentation for that algebra, the underlying variety structure on GL_N is canonical. Likewise, that the multiplication of GL_N is a morphism of variety follows either from the fact that the multiplication of $\mathrm{GL}_N(R)$ is natural in R, or by observing that it is induced by the coproduct on $\mathcal{O}(GL_N)$ given by

$$\Delta(x_{ij}) = \sum_k x_{ik} \otimes x_{ki}.$$

It's well-known that $\mathrm{Rep}\,\mathrm{GL}_N$ is semi-simple, i.e. that GL_N is reductive.

From now on we assume that G is reductive. An algebraic action of G on a variety Z is by definition an action of G on the set Z such that the action map

$$Z \times G \longrightarrow Z$$

is a morphism of variety, i.e. is induced by an algebra map

$$\mathcal{O}(G) \longrightarrow \mathcal{O}(G) \otimes \mathcal{O}(Z).$$

Hence the data of an action of G on Z is the same as the data of a G-representation on $\mathcal{O}(Z)$, making it an object in $\mathrm{Rep}\,G$, such that the multiplication

$$\mathcal{O}(Z)^{\otimes 2} \longrightarrow \mathcal{O}(Z)$$

and the unit

$$\mathbb{C} \longrightarrow \mathcal{O}(Z)$$

are morphisms of G-representations. A fancier way to state this, which will be important for us later, is that $\mathcal{O}(Z)$ is an algebra object (we'll just say "algebra") in the monoidal category $\mathrm{Rep}\,G$.

Definition 1.6. *Let Z be a variety with an algebraic action of G. The categorical quotient $Z//G$ is the variety associated with the algebra $\mathcal{O}(Z)^G$.*

A classical result of Hilbert and Nagata states this algebra is finitely generated provided G is reductive, hence this is well-defined.

Observe that the action of G on itself by conjugation is algebraic, making $\mathcal{O}(G)$ an object in $\mathrm{Rep}\,G$. Let V be a finite-dimensional representation of G and let $v \in V$, $\mu \in V^*$. To this data we attach a function $f_{v,\mu}$ on G given by

$$g \longmapsto \mu(g \cdot v).$$

More invariantly, this maps is nothing but the image of the coaction on V via the canonical isomorphism

$$\mathrm{Hom}_G(V, \mathcal{O}(G) \otimes V) \cong \mathrm{Hom}_G(V \otimes V^*, \mathcal{O}(G)).$$

This induces a map of G-representation

$$i_v : V \otimes V^* \longrightarrow \mathcal{O}(G).$$

Multiplication of those functions is given by the tensor product of representations in the following sense:

$$f_{v,\mu} f_{w,\nu} = f_{v \otimes w, \mu \otimes \nu}.$$

Moreover, if $\phi : V \longrightarrow W$ is a morphism of G-representations, then if $\mu \in W^*$ then

$$f_{\phi(v),\mu} = f_{v,\phi^*(\mu)}$$

where ϕ^* is the transposed map. In particular, this construction "distributes over direct sums", and we get:

Theorem 1.7 (Peter-Weyl decomposition). *For G reductive, the collection of maps $(v,\mu) \mapsto f_{v,\mu}$ factors through an isomorphism in* $\mathrm{Rep}\, G$

$$\bigoplus_{i\in I} V_i \otimes V_i^* \simeq \mathcal{O}(G)$$

where I is a choice of a set of isomorphism classes of irreducible finite dimensional G-representations. Under this identification, the multiplication on $\mathcal{O}(G)$ is given by the canonical map

$$\mathcal{O}(G)^{\otimes 2} \supset (V \otimes V^*) \otimes (W \otimes W^*) \simeq (V \otimes W) \otimes (V \otimes W)^* \longrightarrow \mathcal{O}(G).$$

A convenient way to describe this multiplication is as follows: the G-equivariant map i_V can be dualized to get an operator

$$L_V \in \mathrm{Hom}_G(\mathbb{C}, \mathcal{O}(G) \otimes V \otimes V^*) \cong (\mathcal{O}(G) \otimes \mathrm{End}(V))^G$$

representing the coaction of $\mathcal{O}(G)$ on V. Explicitly, again for $G = \mathrm{GL}_N$ and $V = \mathbb{C}^N$,

$$L_V = \sum_{i,j} x_{ij} \otimes E_{ij}$$

where E_{ij} is the elementary matrix. Observe that $(\mathcal{O}(G) \otimes \mathrm{End}(V))^G$ is an algebra, and the Peter–Weyl description of the multiplication of $\mathcal{O}(G)$ implies the following relation:

$$L_{V\otimes W} = L_V^{(1)} L_W^{(2)} = L_W^{(2)} L_V^{(1)} \tag{1.1}$$

in $(\mathcal{O}(G) \otimes \mathrm{End}(V \otimes W))^G$, where V, W are arbitrary finite-dimensional modules and where

$$L_V^{(1)} = L_V \otimes \mathrm{id}_W \qquad\qquad L_W^{(2)} = \sigma_{W,V}(L_W \otimes \mathrm{id}_V)\sigma_{V,W}$$

and σ is the symmetry of $\mathrm{Rep}\, G$, i.e. the flip of 2-tensors. In that case this really is a complicated way to say that the x_{ij}'s commute, but we'll use a similar equation as a definition for a quantum version of $\mathcal{O}(G)$ in Section 2.4.

1.2. **Character varieties of surfaces.**

Let Γ be a finitely generated (discrete) group.

Proposition 1.8. *To Γ is canonically associated a variety $\mathcal{R}_\Gamma$ whose R-points are*

$$\mathcal{R}_\Gamma(R) = \{\rho : \Gamma \longrightarrow G(R)\}$$

the set of representations of Γ into $G(R)$.

Proof. Fix a presentation

$$\Gamma = \langle g_1, \dots, g_n | w_i \rangle$$

where the w_i are words in the g_i's and their inverses. Let

$$X_k = (x_{ij}^{(k)}), k = 1 \dots n$$

be matrices of variables and set

$$\mathcal{O}(\mathcal{R}_\Gamma) = \mathbb{C}[X_k, \det(X_k)^{-1}]/I$$

where I is the ideal generated by the coefficients of the matrices

$$w_i(X_1, \dots, X_n) - \mathrm{id}$$

where the notation means we substitute $g_k^{\pm 1} \mapsto X_k^{\pm 1}$ (note that X_k^{-1} is a well-defined matrix whose coefficients are polynomials in $x_{ij}^{(k)}$ and $\det(X_k)^{-1}$). For any commutative algebra R there is a natural bijection

$$\mathcal{R}_\Gamma(R) \simeq \mathrm{Hom}_{\mathrm{alg}}(\mathcal{O}(\mathcal{R}_\Gamma), G(R))$$

as required. Along the way, this shows $\mathcal{O}(\mathcal{R}_\Gamma)$ is canonically independent of the chosen presentation of Γ. □

We call $\mathcal{R}_\Gamma$ the $(G\text{-})$representation variety of Γ.

Definition 1.9. *The $(G\text{-})$character variety of Γ is the categorical quotient*

$$\mathrm{Ch}_\Gamma := \mathcal{R}_\Gamma // G.$$

The name "character variety" is justified by the following:

Theorem 1.10 (Artin-Voigt)**.** *Let $G = \mathrm{GL}_N$. There are canonical bijections between the following sets:*

(a) the $\mathbb{C}$-points of the character variety Ch_Γ
(b) the set of equivalence classes of semisimple *N-dimensional representations of Γ.*
(c) the set of characters of N-dimensional representations of Γ

Let S be a compact, connected, oriented surface.

Definition 1.11. *The representation variety of S w.r.t to a basepoint x is*

$$\mathcal{R}_{S,x} := \mathcal{R}_{\pi_1(S,x)}$$

and the character variety of S is

$$\mathrm{Ch}_S := \mathrm{Ch}_{\pi_1(S,x)}\,.$$

Remark 1.12. Since we modded out by the action of G by conjugation, Ch_S is canonically independent of the choice of a basepoint, hence this is well-defined.

Let $S = S_{g,n}$ be a surface of genus g with n boundary components. If $n \geq 1$ then $\pi_1(S)$ is free and a choice of free generators gives an isomorphism of varieties

$$\mathrm{Ch}_S \simeq G^{2g+n-1} // G$$

where G acts by simultaneous conjugation. If $S = S_g$ is closed, let S° be S with a disc removed. The choice of a basepoint x on ∂S°determines a map

$$\mu : \mathcal{R}_{S^\circ,x} \longrightarrow \mathcal{R}_{S^1,x}$$

which, after choosing free generators for $\pi_1(S^\circ, x)$, is given explicitly by

$$(A_1, \ldots, A_g, B_1, \ldots, B_g) \longmapsto \prod_{i=1}^{g} A_i B_i A_i^{-1} B_i^{-1}$$

(the monodromy around the removed disc) so that

$$\mathrm{Ch}_S = \mu^{-1}(\{1\}) // G.$$

Remark 1.13. The affine scheme $\mathrm{Ch}(S)$ is reduced, i.e. the algebra $\mathcal{O}(\mathrm{Ch}(S))$ does not contain any nilpotent element. If $n \geq 1$ this follows from the classical fact that G is. For $n = 0$ this is much harder: for $g > 1$ this is due to Simpson [Sim94, Thm 11.1], and for $g = 1$ this follows from [GG06, Thm 1.2.1].

The following fundamental structure on character varieties of surfaces has been discovered by Atiyah–Bott in the framework of 2d Yang–Mills theory [AB83].

Theorem 1.14. *Let $\mathfrak{g}$ be the Lie algebra of G and b a non-degenerate, symmetric,* ad*-invariant bilinear form on $\mathfrak{g}$. To this data is canonically attached a Poisson structure on* Ch_S.

Remark 1.15. Such a pairing always exists for a reductive G. For $G = \mathrm{GL}_N$ a standard choice is the pairing

$$(A, B) \longmapsto \mathrm{Tr}(AB).$$

Remark 1.16. There are two well-known, equivalent combinatorial models for this Poisson structure due to Fock–Rosly [FR99] and Alekseev–Kosmann-Schwarzbach–Malkin–Meinrenken [AKSM02, AMM98].

1.3. GL_N**-character varieties and skein theory.** In this section we show that functions on the character variety for GL_N, the algebra of observables of our theory, are given by "Wilson loops". Let

$$\mathcal{L}(S) = H_1(LS, \mathbb{C})$$

where LS is the free loop space of S. In other words, $\mathcal{L}(S)$ is the vector space with a basis given by homotopy classes of free loops on S.

Definition 1.17. *The Goldman algebra is*

$$\mathcal{G}(S) = S(\mathcal{L}(S)).$$

It can be thought of as the vector space with a basis given by multi-loops, i.e. (possibly empty) unions of free loops on S. Multiplication is given by union, and the unit is the empty loop.

Proposition 1.18. *There is an evaluation morphism*

$$\mathrm{ev} : \mathcal{L}(S) \longrightarrow \mathcal{O}(\mathrm{Ch}_S)$$

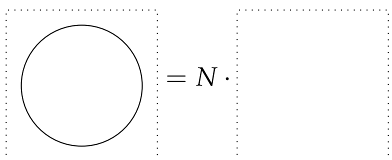
which maps a free loop γ to the function f_γ constructed as follows: let x be an arbitrary point on γ, and let $\tilde{\gamma}$ be the corresponding loop based at x. Let $[\rho] \in \mathrm{Ch}_S$ and choose a lift $\rho \in \mathcal{R}_{\pi_1(S,x)}$. Then we set

$$f_\gamma([\rho]) = \mathrm{Tr}(\rho(\gamma)).$$

One easily see that this is well-defined (i.e. it does not depend on the choices we made), and it extends to an algebra map

$$\mathrm{ev} : \mathcal{G}(S) \longrightarrow \mathcal{O}(\mathrm{Ch}_S).$$

Theorem 1.19 (Procesi [Pro76], Sikora [Sik01])**.** *The map* ev *is surjective and its kernel is generated by the following skein relations:*



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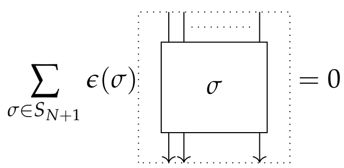
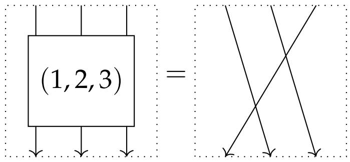


Here, the rectangle labelled by σ means one replaces that rectangle by the obvious picture representing σ, e.g.



The meaning of these relations is that we assume that we have linear combinations of multi-loops on both side, which are all identical (up to homotopy) outside the dotted square. Informally, those relations should be thought of as some sort of abstract characterization of the vector space V of dimension N. The first relation, indeed, literally says that: by definition, a contractible loop on S represents the function

$$\rho \mapsto \mathrm{Tr}(\rho(1)) = \mathrm{Tr}(\mathrm{id}_V) = \dim(V) = N.$$

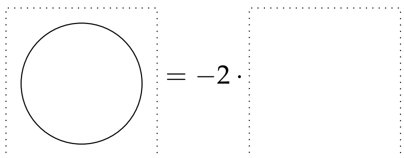
The second relation is a diagrammatic representation of the fact that

$$\bigwedge^{N+1} V \cong (0).$$

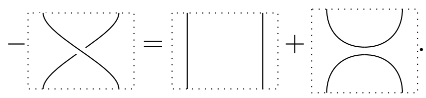
Remark 1.20. There are similar descriptions of functions on the character variety for other groups. You might be familiar with its version for SL_2 which is related to the so-called Kauffman bracket in knot theory. Indeed, if $A \in \mathrm{SL}_2$ then $\mathrm{Tr}(A) = \mathrm{Tr}(A^{-1})$ which translates into the fact that we can represent functions on the character varieties using *unoriented* loops on S. It turns out that the skein relations in that case are nicer if one associates to a loop γ the function

$$\rho \longmapsto -\mathrm{Tr}(\rho(\gamma))$$

(note the minus sign). The kernel of the evaluation map in that situation is generated by

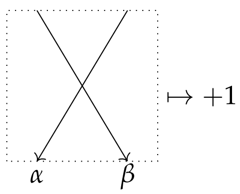


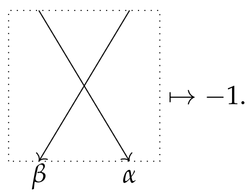
and



Note that this second relation would not make sense for oriented loops ! See the marvelous paper [Tin15] for an explanation of the sign issue.

1.4. **The Goldman Poisson bracket.** Let α, β be two free loops on S in generic position, which in particular means α and β have only finitely many intersection points which are all transverse. Let $p \in \alpha \cap \beta$ and define $\epsilon(\alpha, \beta, p)$ to be $+1$ if the vectors tangent at p to α, β are positively oriented, and -1 otherwise. I.e. assuming the plane is oriented counter-clockwise then





Definition 1.21. *The Goldman bracket of α and β is defined by*

$$\sum_{p\in\alpha\cap\beta} \epsilon(\alpha,\beta,p)\overline{\alpha_p\beta_p}.$$

where α_p, β_p are the loops based at p obtained from α, β respectively.

Theorem 1.22 (Goldman [Gol86]). *The bracket is well-defined and induces on $\mathcal{L}(S)$ the structure of a Lie algebra, which uniquely extends to a Poisson structure on $\mathcal{G}(S)$. The skein relations are compatible with the bracket, hence this descends to a Poisson structure on $\mathcal{O}(\mathrm{Ch}_S)$ which coincides with the one originally defined by Atiyah–Bott.*

Example 1.23. In general, a pair of pants and a punctured torus have the same fundamental group, hence the same character variety. However, for $\mathrm{GL}_1 = \mathbb{C}^\times$ the Goldman bracket for the pair of pants is zero, while for a punctured torus this is the unique bracket on

$$\mathcal{O}(\mathrm{Ch}(T^2\backslash D^2)) \cong \mathbb{C}[x^{\pm1}, y^{\pm1}]$$

such that

$$\{x,y\} = xy.$$

1.5. **Character stacks and skein categories.** It's clear from the definition that the function associated with a loop can be broken down as a composition of maps associated with pieces of that loop, i.e. we want to extend our picture to encompass paths on S and not just loops. Also, the construction of $\mathcal{O}(\mathrm{Ch}_S)$ is not compatible with the gluing of surfaces. Indeed, if S is obtained by gluing two surfaces S', S'' along some part of their boundaries, then there is a map

$$\mathcal{O}(\mathrm{Ch}_{S'}) \otimes \mathcal{O}(\mathrm{Ch}_{S''}) \longrightarrow \mathcal{O}(\mathrm{Ch}_S),$$

which is however not surjective in general.

A fundamental observation [BZFN10] is that one should instead work with the category of quasi-coherent sheaves on the character variety or, more accurately, on the character *stack* $\underline{\mathrm{Ch}}(S)$. We won't need a formal definition of what a stack is, but roughly speaking we have a categorified version of the characterization of the algebra of functions over an affine scheme: $\mathrm{QCoh}(\underline{\mathrm{Ch}}(S))$ is a symmetric monoidal category, such that there are natural equivalences for any commutative algebra R

$$\mathrm{Hom}_\otimes(R\text{-mod}, \mathrm{QCoh}(\underline{\mathrm{Ch}}(S)))^\times \simeq \mathrm{Hom}_{Gpd}(\Pi_1(S), G(R))$$

where the left hand side is the groupoid of colimit preserving symmetric monoidal functor and symmetric monoidal natural isomorphism, and on the right hand side $G(R)$ is seen as a groupoid with one object. For our purpose however, the following working definition will be sufficient:

Definition-Proposition 1.24. *Let X be an affine algebraic variety with algebra of functions $\mathcal{O}(X)$. The category of quasi-coherent sheaves on X is*

$$\mathrm{QCoh}(X) := \mathcal{O}(X)\text{-mod}.$$

If G acts on X algebraically, then the category of quasi-coherent sheaves on the so-called quotient stack X/G is the category

$$\mathrm{QCoh}(X/G) := \mathcal{O}(X)\text{-mod}_G$$

of G-equivariant $\mathcal{O}(X)$-modules. In particular, any choice of a basepoint x on a connected surfaces S induces an equivalence

$$\mathrm{QCoh}(\underline{\mathrm{Ch}}(S)) \simeq \mathcal{O}(\mathcal{R}_{S,x})\text{-mod}_G.$$

A crucial observation is that the character stack of a disc is the quotient of a point by G: we call this the classifying stack $BG = pt/G$ of G. Then the definition above implies that

$$\mathrm{QCoh}(BG) = \mathbb{C}\text{-mod}_G = \mathrm{Rep}\, G.$$

In other words, although the character variety of a disc is trivial, and in particular doesn't know anything about G, the character stack of the disc is interesting and one of the main objectives of these notes is to explain and generalize a precise sens in which the character stack of an arbitrary surface is obtained by gluing together copies of BG.

We also note that, in general, $\mathcal{O}(X)$ is a G-equivariant module over itself the obvious way. This is the structure sheaf of the quotient stack X/G and we have, as algebras,

$$\mathrm{End}_{\mathrm{QCoh}(X/G)}(\mathcal{O}(X)) = \mathcal{O}(X)^G = \mathcal{O}(X//G).$$

More generally, for any $V \in \mathrm{Rep}\, G$ one can consider the free equivariant module $\mathcal{O}(X) \otimes V$. We have a G-equivariant version of the free/forget adjunction

$$\mathrm{End}_{\mathrm{QCoh}(X/G)}(\mathcal{O}(X) \otimes V) \cong \mathrm{Hom}_G(V, \mathcal{O}(X) \otimes V)$$

and of the tensor/hom adjunction

$$\mathrm{Hom}_G(V, \mathcal{O}(X) \otimes V) \cong \mathrm{Hom}_G(\mathbb{C}, \mathcal{O}(X) \otimes \mathrm{End}(V)) \cong (\mathcal{O}(X) \otimes \mathrm{End}(V))^G$$

where $\mathrm{End}(V)$ is seen as a G-module by conjugation.

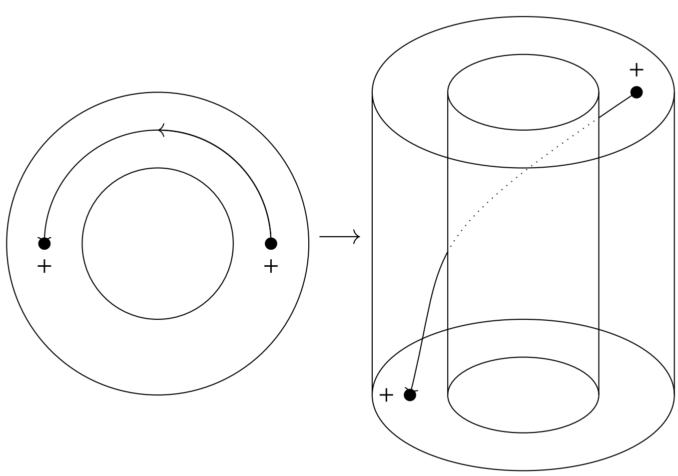
In this section, we'll use this observation in order to show that $\mathrm{QCoh}(\underline{\mathrm{Ch}}(S))$ has a skein theoretic description as well.

Definition 1.25. *The skein category $\mathrm{Sk}_N(S)$ of S has:*

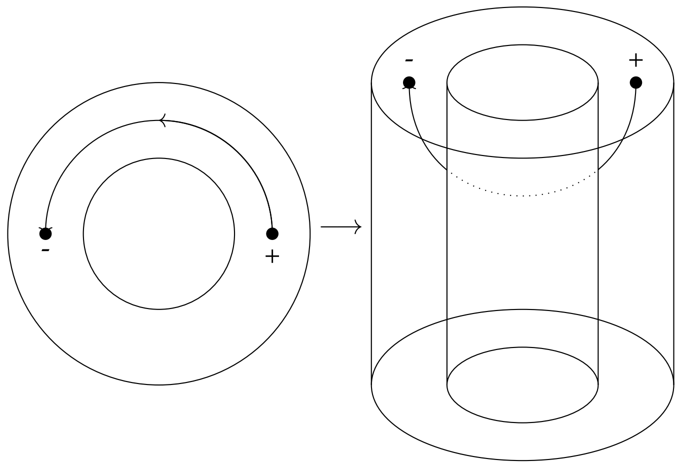
- *objects finite configurations of distinct points on S, labelled by $\{+, -\}$*
- *morphisms formal linear combinations of homotopy classes of multi-paths, such that the start (resp. the end) of each path is labelled by $+$ if the corresponding point belong to the source of the morphism and by $-$ otherwise (resp. vice versa), modulo the skein relations of Theorem 1.19.*

This is well-defined precisely because the skein relations are local: they do not involve the global topology of the surface but only happen inside an embedded disc.

Note that a single path on S can be interpreted as a morphism in $\mathrm{Sk}_N(S)$ in different ways, since each of its endpoints can be chosen to be either in the source or in the target of the corresponding morphism. This is exactly why we need this extra labelling by $\{+, -\}$. To remove this ambiguity, and in anticipation of when we'll replace paths on S by tangles we represent those as being embedded in the cylinder $S \times I$, a morphism from the object at the top to the object at the bottom. E.g.



while



In particular, hom spaces in this category are in fact determined by those between objects only labelled by $+$, and since a path has exactly two endpoints those are 0 unless the number of points at the top and the bottom is the same. In other words, it is enough to understand $\mathrm{End}_{\mathrm{Sk}_N(S)}(++\cdots+)$ where $++\cdots+$ is some fixed choice of a configuration of n points all labelled by $+$.

Hence, let D^2 be the standard unit disc and choose once and for all for each $n \geq 0$ a configuration of points on the x axis inside D^2. Let $\widetilde{\mathrm{Sk}}_N(D^2)$ be the full subcategory of $\mathrm{Sk}_N(D^2)$ which has these configurations, with all possible labellings, as objects. Clearly the inclusion $\widetilde{\mathrm{Sk}}_N(D^2) \subset \mathrm{Sk}_N(D^2)$ is an equivalence of categories.

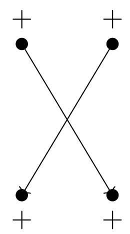
Proposition 1.26. *There is a fully faithful functor*

$$\widetilde{\mathrm{Sk}}_N(D^2) \longrightarrow \mathrm{Rep}\,\mathrm{GL}_N$$

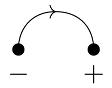
which maps a labelling of the chosen configuration of n points to $\bigotimes_{i=1}^n V^{\epsilon_i}$ *where* $V = \mathbb{C}^N$ *is the defining representation,* ϵ_i *is the labelling of the ith point and*

$$V^+ := V \qquad\qquad V^- := V^*.$$

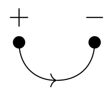
It is defined on elementary morphisms in the following way



$$\mapsto \sigma : \begin{array}{ccc} V \otimes V & \mapsto & V \otimes V \\ u \otimes v & \mapsto & v \otimes u \end{array}$$



$$\mapsto \mathrm{coev} : \mathbb{C} \longrightarrow V^* \otimes V$$



$$\mapsto \mathrm{ev} : V \otimes V^* \longrightarrow \mathbb{C}.$$

Proof. This proposition is basically a reformulation of Schur–Weyl duality: the natural algebra map

$$\mathbb{C}[S_n] \longrightarrow \mathrm{End}_{\mathrm{GL}_N}(V^{\otimes n})$$

is surjective, its kernel is empty if $n \leq N$ and is generated as an ideal by $\sum_{\sigma \in S_{N+1}} \epsilon(\sigma)\sigma$ otherwise, where S_{N+1} acts on, say, the first $N+1$ factors. $\square$

It is well-known that every finite-dimensional GL_N-module can be realized a sub-module (equivalently a quotient) of $V^{\otimes k} \otimes (V^*)^{\otimes l}$ for some integers k, l. Putting this together, one gets

Proposition 1.27. *The category* $\mathrm{Rep}\,\mathrm{GL}_N$*, aka* $\mathrm{QCoh}(\underline{\mathrm{Ch}}(D^2))$*, is equivalent to the category obtained from* $\mathrm{Sk}_N(D^2)$ *by formally adding kernels of idempotent and (possibly infinite) direct sums.*

Now let S be an arbitrary (connected) surface. Choose an embedding of the disc $i : D^2 \hookrightarrow S$ and let $\mathrm{Sk}_N(S, i)$ the full subcategory of $\mathrm{Sk}_N(S)$ whose objects are the configurations of points inside the image of D^2 that we fixed earlier. Again the inclusion $\mathrm{Sk}_N(S, i) \subset \mathrm{Sk}_N(S)$ is an equivalence. Also, the construction of the skein category is clearly functorial with respect to smooth embeddings, so we get a functor

$$\widetilde{\mathrm{Sk}}_N(D^2) \longrightarrow \mathrm{Sk}_N(S, i).$$

Clearly, morphisms in $\mathrm{Sk}_N(S, i)$ can be written as (linear combinations of) compositions of morphisms happening inside the disc, i.e. images of morphisms in $\widetilde{\mathrm{Sk}}_N(D^2)$, and loops on S which are either free of based at one of the chosen points inside of D^2.

Consider the case of a configuration made of a single point x labelled by $+$. We've seen that free loops on S represents functions on the character variety, so what about based loop ? Well, to $\gamma \in \pi_1(S, x)$ we attach the function that takes a genuine representation $\rho \in \mathcal{R}_{S,x}$ (as opposed to an equivalence class) and return $\rho(\gamma)$ acting on $V = \mathbb{C}^N$. This map $\rho \mapsto \rho(\gamma)_{|V}$ is polynomial and G-equivariant. In other words, we get an algebra map

$$\mathbb{C}[\pi_1(S, x)] \longrightarrow (\mathcal{O}(\mathcal{R}_{(S,x)}) \otimes \mathrm{End}(V))^{\mathrm{GL}_N}.$$

More generally, endomorphisms of $(x, +)$ in the skein category can also have free loops components. Indeed, by definition these endomorphisms are represented by linear combination of multi-loops having exactly one component that is based at x and all others being free. We get this way a based version of the Goldman algebra

$$\mathcal{G}(S, x) := \mathcal{G}(S) \otimes \mathbb{C}[\pi_1(S, x)].$$

All in all we again get an evaluation map

$$\mathcal{G}(S,x) \longrightarrow (\mathcal{O}(\mathcal{R}_{S,x}) \otimes \operatorname{End}(V))^{\mathrm{GL}_N}$$

which can be shown again to be surjective and with kernel given by the skein relations.

Hence, we have sketched the proof of the following:

Proposition 1.28. *Any choice of a basepoint x on S gives an equivalence between the skein category $\mathrm{Sk}_N(S)$ and the full sub-category of*

$$\mathrm{QCoh}(\underline{\mathrm{Ch}}(S)) \simeq \mathcal{O}(\mathcal{R}_{S,x})\text{-mod}_{\mathrm{GL}_N}$$

of free equivariant modules of the form $\mathcal{O}(\mathcal{R}_{S,x}) \otimes \bigotimes_{i=1}^n V^{\epsilon_i}$.

Note that, by definition, endomorphisms of the empty configuration on S in the skein category are linear combination of free multiloops on S modulo the skein relation. The functor above maps this to $\mathcal{O}(\mathcal{R}_{S,x})$ as a G-equivariant module over itself, and since this functor is fully faithful we get an algebra isomorphism

$$\operatorname{End}(\emptyset) \cong \mathcal{O}(\mathcal{R}_{S,x})^G,$$

recovering Theorem 1.19 as a particular case.

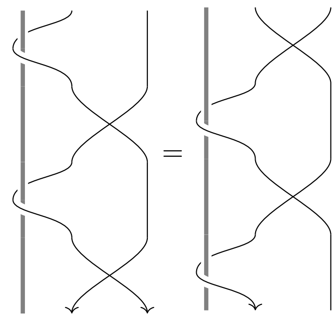
Let us look more closely at the case where $S = S^1 \times [0,1]$ is an annulus. Evaluation on the generator of $\pi_1(S,x) = \mathbb{Z}$ gives an isomorphism of algebraic variety

$$\mathcal{R}_{S,x} \cong \mathrm{GL}_N$$

so that the corresponding function $\mathrm{GL}_N \to \operatorname{End}(V)$ is nothing but the action of GL_N on the module V. In other words, the image of that generator in the skein category is the element in

$$\operatorname{End}_{\mathcal{O}(\mathrm{GL}_N)\text{-mod}_G}(\mathcal{O}(\mathrm{GL}_N) \otimes V) \cong (\mathcal{O}(\mathrm{GL}_N) \otimes \operatorname{End}(V))^G$$

representing the coaction of $\mathcal{O}(G)$ on V, that is the operator L_V. The topological meaning of relation 1.1 is then the following:



where the thick gray strand represents the "inner tube" of the solid cylinder.

2. Categorified linear algebra

2.1. **Locally finitely presentable categories.** In this section we setup the categorical machinery to define and compute factorization homology with coefficient in a braided tensor category. The very first thing we need to do is to choose a target, formally a symmetric monoidal (2,1)-category of linear categories. There a many possible choices here, but the previous section showed that we need to consider large categories like $\operatorname{Rep} G$, the category of all $\mathcal{O}(G)$-comodules, basically because we went $\mathcal{O}(G)$ itself to be an object in there. A very convenient framework to talk about categories of modules, comodules and more generally of categories of quasi-coherent sheaves, is that of locally finitely presentable category. This requires some formal definitions. For a complete reference on that topic, see [AR94].

Let $\mathcal{C}$ be a $\mathbb{C}$-linear category.

Definition 2.1. *An object $x \in \mathcal{C}$ is called* compact *(or sometimes* finitely presentable*) if the functor*

$$\mathrm{Hom}(x, -) : \mathcal{C} \longrightarrow \mathrm{Vect}_{\mathbb{C}}$$

commutes with filtered colimits. It is called tiny*, or* compact-projective *if* $\mathrm{Hom}(x, -)$ *commutes with all small colimits.*

Remark 2.2. In the linear case, an object is tiny if and only if it is simultaneously compact and projective, hence the name.

Let $\mathcal{C}_c$ be the full sub-category of compact objects.

Definition 2.3. *The category $\mathcal{C}$ is called* locally finitely presentable *if it is cocomplete, i.e. it has all small colimits, and if there is an essentially small collection of compact objects $S \subset \mathcal{C}_c$ which is a strong generator, meaning that every object in $\mathcal{C}$ can be written as a colimit of objects in S. It is called* locally tiny *if there exists such a S made of tiny objects.*

Remark 2.4. Another characterization of S being a strong generator is that the collection of functors $\{\mathrm{Hom}(c, -)\}_{c \in S}$ is jointly conservative, i.e. a morphism $f : x \to y$ is an isomorphism if and only if all of the induced maps

$$\mathrm{Hom}(c, x) \longrightarrow \mathrm{Hom}(c, y)$$

are.

Remark 2.5. It's easily seen that a finite colimit of compact objects is compact ([AR94, Prop. 1.3]) and, in fact, if $\mathcal{C}$ is LFP then $\mathcal{C}_c$ is the closure of S under finite colimits in $\mathcal{C}$, and in particular is essentially small, so that $\mathcal{C}_c$ is also a strong generator. It also means that one can always find a possibly larger set $S \subset \bar{S} \subset \mathcal{C}_c$ such that every object is a *filtered* colimit of objects in $\bar{S}$ (e.g. one can take $\bar{S} = \mathcal{C}_c$, but often there are smaller choices)[2]. In case $S = \mathcal{C}_c$ every object x is in fact *canonically* a filtered colimit of objects in $\mathcal{C}_c$ [AR94, Thm 1.1, Prop 1.22], the colimit over the forgetful functor

$$D_x : \mathcal{C}_c \downarrow x \longrightarrow \mathcal{C}$$

where the slice category $\mathcal{C}_c \downarrow x$ has objects maps $c \to x, c \in \mathcal{C}_c$ and morphisms commutative diagrams. Likewise, in a locally tiny category, every object is canonically a colimit, however not necessarily filtered in that case, of tiny objects.

Example 2.6. The typical examples of locally finitely presentable categories to have in mind are:

- The category A-mod for an algebra A: compact objects are finitely presentable A-modules, hence the name. This category is in fact locally tiny, in which case two examples of generators are given by the collection of finitely generated projective (resp. free) modules.
- The category C-comod for a coalgebra C: compact objects are *finite-dimensional* C-comodules. In that case, that this category is locally finitely presentable is basically a reformulation of the fundamental theorem of comodules (Thm 1.2). This category is not always locally tiny: categories of comodules may fail to have enough projective objects.

[2] Here were are using that a category closed under finite colimits is automatically a filtered category

Those categories are naturally the objects of a 2-category, but we'll actually need something a little bit simpler:

Definition 2.7. *A* $(2,1)$*-category is a 2-category in which every* 2*-morphism is invertible. In particular, a strict (2,1)-category is the same as a category enriched in groupoids.*

Remark 2.8. Typical examples of (strict) (2,1)-categories arise as follows: let $\mathcal{C}$ be a category enriched in topological spaces, i.e. hom spaces are topological spaces and compositions maps are continuous. Then we can define a (2,1)-category $h^2\mathcal{C}$ with the same objects as $\mathcal{C}$, and morphisms

$$\mathrm{Hom}_{h^2\mathcal{C}}(X,Y) := \Pi_1(\mathrm{Hom}_{\mathcal{C}}(X,Y))$$

where Π_1 is the fundamental groupoid.

Definition 2.9. *We denote by* LFP *the* $(2,1)$*-category whose objects are locally finitely presentable categories, morphisms are linear functors which are cocontinuous, i.e. which commute with small colimits, and 2-morphisms are natural* iso*morphisms. We denote by* LFP_c *the* $(2,1)$*-category in which we additionally require functors to preserve compact objects.*

One should think of LFP categories as some kind of categorified vector spaces, and of a choice of a generator as some kind of basis. The following Proposition thus says that "a linear map is determined by what it does to a basis", i.e. that $\mathcal{C}$ is in fact *freely* generated under filtered colimits by $\mathcal{C}_c$.

Proposition 2.10. *Let* $\mathcal{C}$, $\mathcal{D}$ *be in* LFP, *then restriction along* $\mathcal{C}_c$ *induces an equivalence between* $\mathrm{Hom}(\mathcal{C},\mathcal{D})$ *and the category of right exact*[3] *linear functors* $\mathcal{C}_c \to \mathcal{D}$.

Crucial for us is the so-called special adjoint functor theorem (SAFT):

Proposition 2.11. *Let* $F : \mathcal{C} \longrightarrow \mathcal{D}$ *be a functor in* LFP. *Then* F *has a right adjoint. Moreover,* F *preserves compact objects if and only if its right adjoint preserves filtered colimits. If* $\mathcal{C}$ *is locally tiny, then* F *preserves tiny objects if and only if its right adjoint is cocontinuous.*

Remark 2.12. Prop. 2.10 implies that LFP_c is in fact equivalent to the perhaps more familiar (2,1)-category REX of essentially small categories which are closed under *finite* colimits, and right exact functors. However, although Prop. 2.11 says that under reasonable assumptions those functors will have right adjoints in LFP, those will rarely be themselves compact preserving, which is the main reason to work with LFP.

2.2. **The Deligne–Kelly tensor product.** Accordingly, we can define a tensor product on LFP by mimicking the vector space definition:

Definition 2.13. *Let* $\mathcal{C},\mathcal{D} \in$ LFP. *A Deligne–Kelly tensor product of* $\mathcal{C}$ *and* $\mathcal{D}$ *is a pair of an* LFP*-category* $\mathcal{C} \boxtimes \mathcal{D}$ *and a bifunctor*

$$\mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{C} \boxtimes \mathcal{D}$$

which is linear and cocontinuous in each variable, and which is universal for this property.

Theorem 2.14 (Kelly, see also [Fra13]). *The Deligne-Kelly tensor product exists and turn* LFP *into a symmetric monoidal (2,1)-category, where the symmetry is induced by the obvious flip, and the unit is* $\mathrm{Vect}_{\mathbb{C}}$. *In fact,* LFP *is even* closed, *i.e.* $\mathrm{Hom}(\mathcal{C},\mathcal{D}) \in$ LFP *and there are natural equivalences*

$$\mathrm{Hom}(\mathcal{C} \boxtimes \mathcal{D}, \mathcal{E}) \simeq \mathrm{Hom}(\mathcal{C}, \mathrm{Hom}(\mathcal{D},\mathcal{E})).$$

[3]We call a functor right exact if it commutes with finite colimits. If the categories at hand are abelian this coincides with the usual definition as a functor which preserves right exact sequences, but this is more general.

Remark 2.15. For an excellent discussion of symmetric monoidal structures on 2-categories, see [SP09, Chap. 2].

Remark 2.16. We denote by $x \boxtimes y$ the image of the pair (x, y) in $\mathcal{C}_c \boxtimes \mathcal{D}_c$ and we call those the pure tensors. We have

$$\mathrm{Hom}_{\mathcal{C}\boxtimes\mathcal{D}}(x \boxtimes y, x' \boxtimes y') \simeq \mathrm{Hom}_{\mathcal{C}}(x, x') \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathcal{D}}(y, y')$$

hence in particular $x \boxtimes y$ is again compact. Those generates $\mathcal{C} \boxtimes \mathcal{D}$ under filtered colimits, and compact objects in $\mathcal{C} \boxtimes \mathcal{D}$ are finite colimits of pure tensors. Together with Proposition 2.10 this implies that a functor out of $\mathcal{C} \boxtimes \mathcal{D}$ is uniquely characterized by what it does to pure tensors, a trick we will use often.

Remark 2.17. The Deligne–Kelly tensor product generalizes the tensor product of algebras in the sense that

$$a\text{-mod} \boxtimes b\text{-mod} \simeq (a \otimes b)\text{-mod}.$$

2.3. **Monoidal, braided and ribbon categories.** In this section we adapt the usual definition of a monoidal category to our setting.

Definition 2.18. *A (strict) monoidal category is an algebra object in* LFP_c*, i.e. a category* $\mathcal{A} \in \mathrm{LFP}$ *equipped with a tensor product*

$$\otimes : \mathcal{A} \boxtimes \mathcal{A} \longrightarrow \mathcal{A}$$

and a unit

$$\mathbf{1}_{\mathcal{A}} \in \mathcal{A}$$

such that $\mathbf{1}_{\mathcal{A}}$ *is compact, and* $\otimes$ *is unital and associative and preserves compact objects.*

A braiding on $\mathcal{A}$ *is a linear natural isomorphism*

$$\beta : x \otimes y \simeq y \otimes x$$

satisfying the usual hexagon axioms.

A balancing on the braided monoidal $\mathcal{A}$ *is a natural automorphism* θ *of the identity which satisfies*

$$\theta_{x \otimes y} = \beta_{y,x}\beta_{x,y}\theta_x\theta_y.$$

Remark 2.19. The technical condition that the tensor product preserves compact objects will be crucial in the proofs, which is why we choose to make it part of the definition.

A fundamental property that monoidal categories can have and which shows up often in representation theory and quantum topology is rigidity. Usually one requires all objects x to have a, right say, dual in $\mathcal{A}$, i.e. another object x^* and maps

$$\mathrm{ev} : x \otimes x^* \longrightarrow \mathbf{1}_{\mathcal{A}}$$

and

$$\mathrm{coev} : \mathbf{1}_{\mathcal{A}} \longrightarrow x^* \otimes x$$

satisfying the usual zig-zag identities. This doesn't quite make sense for our large categories, e.g. $\mathrm{Vect}_{\mathbb{C}}$, the category of all vector spaces, is clearly not rigid in that sense. The natural thing to ask in our setting is instead the following:

Definition 2.20. *A monoidal category is rigid if all* compact *objects have a left and right dual. If* $\mathcal{A}$ *is braided and balanced, we say that* $\mathcal{A}$ *is ribbon if it is rigid and if* $\theta_{x^*} = \theta_x^*$.

Example 2.21. Let H be an Hopf algebra. Then the monoidal category H-comod is rigid in that sense, but crucially the monoidal category H-mod is *not*.

Rigidity is a strong property which has some interesting consequences, e.g.:

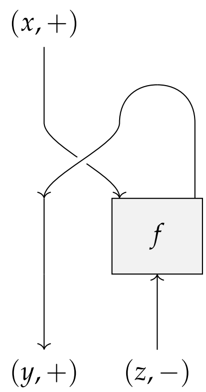
- The condition that we impose that the tensor product of $\mathcal{A}$ should preserve compact objects is in fact automatic if $\mathcal{A}$ is rigid, and this is usually how we verify this condition in practice: if x, y are compact, then
$$\mathrm{Hom}(x \otimes y, -) \cong \mathrm{Hom}(y, x^* \otimes -)$$
and since by assumption y is compact, and $\otimes$ preserves colimits in each variable, $x \otimes y$ is compact as well. If y was in fact tiny, then this show $x \otimes y$ is as well: in other words the sub-category of tiny objects is a "tensor ideal" in that case.
- If $\mathcal{A}$ is rigid and if the unit $\mathbf{1}_{\mathcal{A}}$ is tiny (i.e. if it's projective, since it's already compact by assumption), then somewhat surprisingly $\mathcal{A}$ is in fact automatically semisimple. Indeed for any compact x,
$$\mathrm{Hom}(x, -) \cong \mathrm{Hom}(\mathbf{1}_{\mathcal{A}}, x^* \otimes -).$$
But the functor on the right hand side commutes with all colimits by assumption, hence x is also projective.

It's well-known that (ordinary) ribbon categories yield links and tangles invariant. We record the following version of the statement:

Definition 2.22. *Let $\mathcal{A}$ be a ribbon category (as defined above) and let $\mathcal{T}_{\mathcal{A}}$ be the category whose:*

- *objects are finite sequences of pairs (x, ϵ) where x is a compact (hence dualizable) object in $\mathcal{A}$ which we think of as sitting on the horizontal axis on the standard disc D^2*
- *morphisms are linear combinations of isotopy classes of decorated ribbon graphs, i.e. framed, oriented graphs embedded in $D^2 \times I$, with edges labelled by compact objects in $\mathcal{A}$ and vertices are labelled by morphisms in $\mathcal{A}$ in a compatible way.*

Here is a typical example of a ribbon graph:



where f is a morphism $x \otimes y^* \to z^*$.

Theorem 2.23 ([RT90, Tur10, Tur90])**.** *There is a unique linear functor (the Reshetikhin–Turaev functor)*
$$\mathrm{RT}_{\mathcal{A}} : \mathcal{T}_{\mathcal{A}} \longrightarrow \mathcal{A}$$
which maps the sequence $(x_1, \epsilon_1) \ldots (x_k, \epsilon_k)$ to the tensor product $x_1^{\epsilon_1} \otimes \ldots \otimes x_k^{\epsilon_k}$, where $x^+ = x$ and $x^- = x^*$, the elementary crossing, the cup and the cap to the braiding, evaluation and coevaluation in $\mathcal{A}$ respectively, and a labelled vertex to the corresponding morphism.

2.4. **The main example: quantum groups.** A good reference for this section is [KS97]. Let q be a generic complex number (in this case this means q is neither 0 nor a non-trivial root of unity). We also fix a dth root $q^{1/d}$ root of q for some integer d depending on G and some normalization choices. The corresponding quantum group is a certain Hopf algebra $\mathrm{Fun}_q(G)$ which is a q-deformation of $\mathcal{O}(G)$ in the sense that $\mathrm{Fun}_1(G) = \mathcal{O}(G)$. For $G = \mathrm{GL}_N$ it can be described as follows:

Let

$$R = q^{1/N}\left(\sum_{i\neq j} E_{ii}\otimes E_{jj} + q\sum E_{ii} + (q-q^{-1})\sum_{i<j} E_{ij}\otimes E_{ji}\right) \in \mathrm{End}(V^{\otimes 2})$$

where $V = \mathbb{C}^N$ (the reason for the factor $q^{1/N}$ will be explained in Remark 4.2). Then R is a solution of the quantum Yang–Baxter equation

$$R^{1,2}R^{1,3}R^{2,3} = R^{2,3}R^{1,3}R^{1,2}$$

in $\mathrm{End}(V^{\otimes 3})$ where $R_{1,2} = R\otimes \mathrm{id}_V$ etc...

Definition 2.24. *The bialgebra $\mathrm{Fun}_q(\mathrm{End}(V))$ is generated by the matrix of symbols $X = (x_{ij})$ and relations*

$$RL_V^{(1)}L_V^{(2)} = L_V^{(2)}L_V^{(1)}R \tag{2.1}$$

where $L_V^{(1)} := \sum x_{ij}\otimes E_{ij}\otimes \mathrm{id}$ and $L_V^{(2)} := \sum x_{ij}\otimes \mathrm{id}\otimes E_{ij}$. This, of course, should be compared with (1.1).

Its coproduct is defined by

$$\Delta(x_{ij}) = \sum x_{ik}\otimes x_{ki}.$$

The bialgebra $\mathrm{Fun}_q(\mathrm{GL}_N)$ is obtained from $\mathrm{Fun}_q(\mathrm{End}(V))$ by inverting the quantum determinant

$$\mathrm{qdet}(X) := \sum_{\sigma\in S_n} q^{l(\sigma)} x_{1\sigma(1)}\dots x_{n\sigma(n)}$$

where l is the length. This bialgebra has an antipode making it an Hopf algebra.

This is called the FRT presentation of $\mathrm{Fun}_q(\mathrm{GL}_N)$ (referring to the authors of [FRT90]), and (2.1) is often called the RLL equation.

Definition 2.25. *Denote by $\mathrm{Rep}_q G$ the category of $\mathrm{Fun}_q(G)$-comodules.*

There is a bilinear pairing r on $\mathrm{Fun}_q(G)$ which for $G = \mathrm{GL}_N$ is induced by

$$r(X\otimes X) := R \in \mathrm{End}(V\otimes V)$$

and a linear form θ on $\mathrm{Fun}_q(G)$ making it a co-ribbon Hopf algebra, meaning that $\mathrm{Rep}_q G$ is a ribbon category. By design, the braiding on $V\otimes V$ for $V = \mathbb{C}^N$ is given by $P\circ R$ where P is the permutation $P(u\otimes v) = v\otimes u$.

We have the following highly non trivial fact, variants of which are due to Drinfeld [Dri87] and Kazhdan–Lusztig [KL94].

Theorem 2.26. *For q generic, $\mathcal{O}(G)$ and $\mathrm{Fun}_q(G)$ are isomorphic* as coalgebras, *i.e. we have a vector space preserving equivalence of linear categories*

$$\mathrm{Rep}\, G \simeq \mathrm{Rep}_q G.$$

2.5. **Module categories and categories of modules.**

Definition 2.27. *Let $\mathcal{A}$ be a monoidal category. A (strict) right $\mathcal{A}$-module is an* LFP *category equipped with a functor in* LFP_c *(i.e. compact preserving)*

$$\triangleleft : \mathcal{M} \boxtimes \mathcal{A} \longrightarrow \mathcal{M}$$

which is associative the obvious way. Left and bi modules are defined in a similar way.

Fundamental examples of that kind of structure arise as follows: let a be an algebra object in $\mathcal{A}$, i.e. a is equipped with morphisms

$$\mu_a : a \otimes a \longrightarrow a$$

and

$$1 : \mathbf{1}_{\mathcal{A}} \longrightarrow a$$

which makes the obvious diagrams expressing that $(a, \mu, 1)$ is associative and unital commute. Likewise, there is a natural notion of, left say, a-module internally to $\mathcal{A}$: this is an object equipped with an action map $\mu_m : a \otimes m \to m$ making the obvious diagram commute.

Proposition 2.28. *The category a-$\mathrm{mod}_{\mathcal{A}}$ of left a-modules in $\mathcal{A}$ is in* LFP*, and is a right $\mathcal{A}$-module with action given by*

$$m \triangleleft x := m \otimes x$$

with a-module structure induced by that of m.

Proof. The fundamental observation is that colimits in $\mathcal{M} = a\text{-}\mathrm{mod}_{\mathcal{A}}$ can be computed in $\mathcal{A}$: let $(m_i)_{i \in I}$ be a diagram in $\mathcal{M}$, and let m be the colimit of this diagram but in $\mathcal{A}$ instead. The crucial assumption that we are using is that the tensor product of $\mathcal{A}$ commutes with colimits. Hence, the a-modules structures

$$a \otimes m_i \longrightarrow m_i$$

assemble into a map

$$\mathrm{colim}_I (a \otimes m_i) \cong a \otimes \mathrm{colim}_I\, m_i = a \otimes m \longrightarrow m$$

which turns m into an a-module. Pretty much by construction, this makes m the colimit of the m_i's in $\mathcal{M}$. Hence, $\mathcal{M}$ is cocomplete and the forgetful functor creates (and in particular preserves) small colimits. This also implies that the action functor

$$\mathcal{M} \times \mathcal{A} \longrightarrow \mathcal{M}$$

is cocontinuous in each variable.

Just like for ordinary algebras, the forgetful functor has a left adjoint mapping $x \in \mathcal{A}$ to the free module $a \otimes x$. If $m \in \mathcal{M}$, then the action $\mu_m : a \otimes m \to m$ of a is a map of a-modules. One then shows that m is the coequalizer of the following diagram:

$$a \otimes a \otimes m \underset{\mu_a \otimes \mathrm{id}_m}{\overset{\mathrm{id}_a \otimes \mu_m}{\rightrightarrows}} a \otimes m.$$

Now if x is compact in $\mathcal{A}$, then $a \otimes x$ is compact in $\mathcal{M}$: indeed the free module functor preserves colimits and has a right adjoint (the forgetful functor) which also does, hence preserves filtered colimits in particular, so that it follows from the SAFT.

All in all, any object in $\mathcal{M}$ can be written as a small colimit of compact objects, namely those of the form $a \otimes x$ for x compact. Hence $\mathcal{M} \in$ LFP, and the action functor descends to a colimit preserving functor

$$\mathcal{A} \boxtimes \mathcal{M} \longrightarrow \mathcal{M}.$$

Finally, any compact object in $\mathcal{M} \boxtimes \mathcal{A}$ is a finite colimit of objects of the form $(a \otimes x) \boxtimes y$ with x, y compact. Since the action functor preserves colimit, the image of any such object in $\mathcal{M}$ is the colimit of objects of the form $a \otimes (x \otimes y)$. Since $x \otimes y$ is compact by our assumption on $\mathcal{A}$, we conclude that the action functor preserves compact objects. □

2.6. **Relative tensor products and traces.** The Deligne–Kelly tensor product can be extended to modules.

Definition 2.29. *Let $\mathcal{A}$ be a monoidal category, and $\mathcal{M}, \mathcal{N}$ be right and left $\mathcal{A}$-modules respectively. A balanced functor is the data of a bifunctor*

$$B : \mathcal{M} \times \mathcal{N} \longrightarrow \mathcal{E}$$

into some target category $\mathcal{E}$, which as usual is cocontinuous and linear in each variable, and of a natural isomorphism

$$B(m \triangleleft x, n) \cong B(m, x \triangleright n)$$

satisfying certain coherence conditions.

The relative tensor product $\mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{N}$ of $\mathcal{M}$ and $\mathcal{N}$ is the universal object among LFP*-categories which are the target of a balanced functor.*

Example 2.30. Let $\mathcal{M} = a\text{-mod}_{\mathcal{A}}$ and $\mathcal{N} = {}_{\mathcal{A}}\text{mod-}b$ be the categories of left and right modules for algebras a, b in $\mathcal{A}$ respectively. Then $\mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{N}$ is equivalent to the category of a-b-bimodules in $\mathcal{A}$.

Slightly more generally, recall that if a is an algebra and m a bimodule, one can consider the trace, aka cocenter, aka 0th Hochschild homology

$$\mathrm{Tr}_a(m) := a/[a, m].$$

In particular, if m, n are right and left a-modules respectively, then $m \otimes n$ is an a-bimodule and

$$\mathrm{Tr}_a(m \otimes n) = m \otimes_a n.$$

This notion can be categorified as well (see e.g. [FSS14]): let $\mathcal{A}$ be monoidal and $\mathcal{M}$ be an $\mathcal{A}$-bimodule. A balanced functor is a functor

$$B : \mathcal{M} \longrightarrow \mathcal{E}$$

into some category, together with a natural isomorphism $B(a \triangleright m) \cong B(m \triangleleft a)$ satisfying certain coherence conditions. Then the trace of $\mathcal{M}$ is a category $\mathrm{Tr}_{\mathcal{A}}(\mathcal{M})$ equipped with a balanced functor $\mathcal{M} \to \mathrm{Tr}_{\mathcal{A}}(\mathcal{M})$, which is universal for this property. Again, one can show that this indeed exists, and that for a right module $\mathcal{M}$ and left module $\mathcal{N}$, then $\mathrm{Tr}_{\mathcal{A}}(\mathcal{M} \boxtimes \mathcal{N}) \simeq \mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{N}$.

3. Factorization homology

3.1. Definition.

Definition 3.1. *Let* Surf *be the strict symmetric monoidal $(2,1)$-category defined as follows:*

- *objects are compact, oriented surfaces, with or without boundary, including the empty surface $\varnothing$.*
- *for $X, Y \in$ Surf, the groupoid $\mathrm{Hom}(X, Y)$ is the fundamental groupoid*

$$\Pi_1(\mathrm{Emb}(X, Y))$$

of the space of smooth oriented embedding of X into Y.

The monoidal structure is given by disjoint union and the unit by $\varnothing$.

We let Disc *be the full subcategory of* Surf *whose objects are disjoint unions of the standard unit disc D^2.*

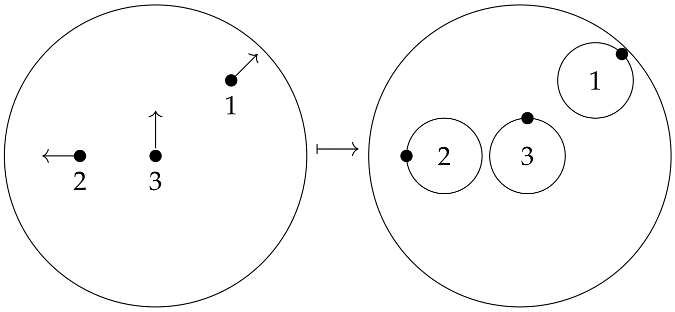
Let $\overrightarrow{\mathrm{Conf}}_n(D^2)$ be the framed configuration space of points in a disc

$$\overrightarrow{\mathrm{Conf}}_n(D^2) = \{((z_1, v_1), \dots, (z_n, v_n)) \in (D^2 \times S^1)^n | i \neq j \Rightarrow z_i \neq z_j\}$$

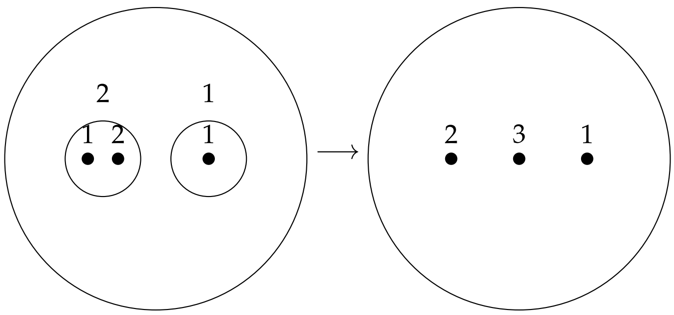
where v_i is a unit tangent vector at z_i. It is easily seen that the continuous map

$$\overrightarrow{\mathrm{Conf}}_n(D^2) \longrightarrow \mathrm{Emb}\left(\bigsqcup_{i=1}^{n} D^2, D^2\right)$$

which sends a framed point (z_i, v_i) to the disc with center z_i and such that the unit vector v_i is pointing toward the image of the point $(1, 0) \in D^2$ is an homotopy equivalence[4].



Just like we did for the skein category, we can choose once and for all for each n a set of $n!$ basepoints for $\overrightarrow{\mathrm{Conf}}_n$ along the horizontal axis corresponding to the possible labelling of a fixed configuration of points by integers $1 \dots n$. We let $\widetilde{\mathrm{Disc}}$ be the full category of Disc where we restrict to the embeddings determined by those basepoints. For any embedding $\bigsqcup_{i=1}^{n} D^2 \hookrightarrow D^2$ and any choice of one of our based configuration of k_i framed points on the ith embedded disc, there is a unique-up-to-homotopy homotopy staying on the horizontal axis from the images of those configurations in the larger disc, to the ordered configuration of $\sum k_i$ points on the larger disc, shifting the labelling the obvious way. This implies that composition in $\widetilde{\mathrm{Disc}}$ is well-defined.



The inclusion $\widetilde{\mathrm{Disc}} \subset \mathrm{Disc}$ is an equivalence of symmetric monoidal $(2, 1)$-categories. The reason for introducing those objects is the following fundamental result. As its core it is essentially a "coordinate-free" definition of balanced braided monoidal categories, or in other words that those are indeed the correct categorical structures carrying compatible representations of the framed braid groups $\pi_1(\overrightarrow{\mathrm{Conf}}_n / S_n)$. In other words this should be the "correct" definition of

[4]Slightly more precisely, after fixing an embedding $\iota : D^2 \hookrightarrow D^2$, precomposing ι with a rotation gives a map $S^1 \simeq SO(2) \to \mathrm{Emb}(D^2, D^2)$ which is an homotopy equivalence. This is why an embedding of $\sqcup_n D^2$ into D^2 is determined, up to a *contractible* space of choices, by the image of the center of each disc and a unit vector based at it.

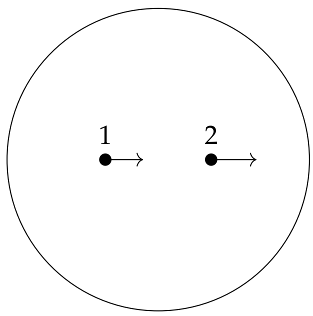
(balanced) braided monoidal category, and the theorem below states that it coincides with the usual axiomatic one. In a slightly different formulation it is due to Deligne (unpublished), Bezrukavnikov–Finkelberg–Schechtman [BFS06], Salvatore–Wahl [SW01]. Note that we state the theorem with LFP_c as a target since this is the setting in which we want to do computations later on, but we have a similar statement for any target.

Observe that the $(2,1)$-category of symmetric monoidal functors $\mathrm{Disc} \to \mathrm{LFP}$ itself has a symmetric monoidal structure given by pointwise multiplication.

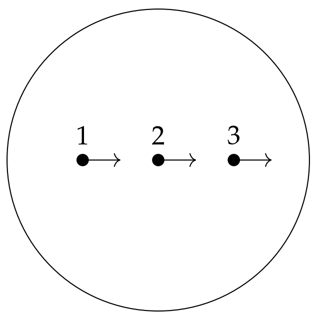
Theorem 3.2. *Evaluation on D^2 induces an equivalence of symmetric monoidal $(2,1)$-categories between*

- *symmetric monoidal functors*
$$\mathrm{Disc} \longrightarrow \mathrm{LFP}_c$$
- *the $(2,1)$-category of balanced braided monoidal categories in LFP_c, braided monoidal functors in LFP_c and natural isomorphisms of those.*

Sketch of proof. Since the inclusion $\widetilde{\mathrm{Disc}} \subset \mathrm{Disc}$ is an equivalence, we can instead look at symmetric monoidal functors
$$\widetilde{\mathrm{Disc}} \longrightarrow \mathrm{LFP}\,.$$
Let $\mathcal{A} \in \mathrm{LFP}$ be the image of D^2. That the functor at hand is monoidal implies that the image of the disjoint union of n discs is $\mathcal{A}^{\boxtimes n}$. In particular $\emptyset$ is mapped to $\mathrm{Vect}_{\mathbb{C}}$. The configuration



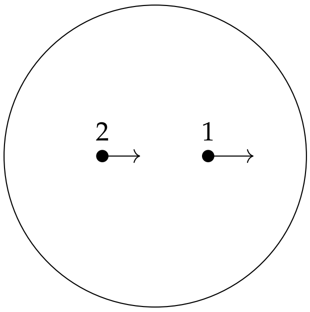
is mapped to a functor
$$\otimes : \mathcal{A}^{\boxtimes 2} \longrightarrow \mathcal{A}.$$
By construction, the configuration



is the image both of $\otimes \circ (\mathrm{Id}_{\mathcal{A}} \boxtimes \otimes)$ and of $\otimes \circ (\otimes \boxtimes \mathrm{Id}_{\mathcal{A}})$ which means that $\otimes$ is associative. The unique inclusion of $\emptyset$ into D^2, i.e. the unique configuration of zero points inside D^2, induces a functor
$$\mathrm{Vect}_{\mathbb{C}} \longrightarrow \mathcal{A}$$

whose value on $\mathbb{C}$ is a compact object $\mathbf{1}_{\mathcal{A}}$ which is easily seen to be a unit for $\otimes$. Therefore, $(\mathcal{A}, \otimes, \mathbf{1}_{\mathcal{A}})$ is a strict monoidal category.

Since the functor at hand is symmetric, the configuration



is mapped to the functor $\otimes^{\mathrm{op}}$ defined as the composition

$$\mathcal{A}^{\boxtimes 2} \xrightarrow{\sigma_{\mathcal{A},\mathcal{A}}} \mathcal{A}^{\boxtimes 2} \xrightarrow{\otimes} \mathcal{A}$$

where σ is the symmetry of LFP. In other words, this is the opposite multiplication in $\mathcal{A}$

$$x \otimes^{\mathrm{op}} y = y \otimes x.$$

The obvious path between those two configurations switching them in a counter-clockwise way induces a natural isomorphism

$$\beta_{x,y} : x \otimes y \cong y \otimes x$$

and the path in $\overrightarrow{\mathrm{Conf}}_1$ which takes our chosen basepoint and makes its unit tangent vector do a full counter clockwise turn induces a natural automorphism θ of the identity functor of $\mathcal{A}$. All in all we see that we obtain on $\mathcal{A}$ the structure of a balanced braided monoidal category. It is clear that any morphism and 2-morphisms in $\widetilde{\mathrm{Disc}}$ are obtained by composition and disjoint union/tensor product of those. What is much less obvious is that we also get all relations, i.e. that a balanced braided structure on $\mathcal{A}$ is sufficient to get such a symmetric monoidal functor, but this is indeed true. $\square$

We are finally in a position where we can define factorization homology with coefficients in a balanced braided tensor category.

Theorem 3.3 ([BD04, Lur09, AF15]). *Let $\mathcal{A}$ be a balanced braided monoidal category seen as a symmetric monoidal functor*

$$\mathrm{Disc} \longrightarrow \mathrm{LFP}\,.$$

There is a universal, canonical extension of $\mathcal{A}$ to a symmetric monoidal functor

$$\begin{tikzcd} \mathrm{Disc} \arrow[r, "\mathcal{A}"] \arrow[d, hook] & \mathrm{LFP} \\ \mathrm{Surf} \arrow[ur, "\int_{(-)} \mathcal{A}"'] & \end{tikzcd}$$

which is characterized by excision, a property explained below. We call the category $\int_S \mathcal{A}$ "factorization homology of S with coefficients in $\mathcal{A}$".

Remark 3.4. As the notation suggests, $\int_S \mathcal{A}$ is obtained by somehow integrating over all embeddings of a bunch of discs into S. Formally, it is the $(2,1)$-colimit over the image through the functor associated with $\mathcal{A}$ of the slice category Disc $\downarrow$ S.

3.2. **Excision.** By construction, the image $\int_{D^2} \mathcal{A}$ of the standard disc is $\mathcal{A}$ as a balanced braided monoidal category. Functoriality w.r.t smooth oriented embeddings implies that if P is a compact 1-dimensional manifold, then $\int_P \mathcal{A} := \int_{P\times I} \mathcal{A}$ has a monoidal structure[5]. Given a collar, i.e. an embedding $P \times I \hookrightarrow S$ restricting to $P \times \{0\} \hookrightarrow \partial S$ (resp. $P \times \{1\} \hookrightarrow \partial S$), $\int_S \mathcal{A}$ becomes a left (resp. right) $\int_{P\times I} \mathcal{A}$-module. With these structure at hand, the following fundamental property of factorization homology, stating that it is a TFT in a very strong sense, is the key to explicit computations:

Theorem 3.5 (Excision)**.** *Let S be a surface with "right" and "left" collars $P \times I$ respectively, and let S_P be the surface obtained by gluing along P (and smoothing if necessary). Then the functor*

$$\int_S \mathcal{A} \longrightarrow \int_{S_P} \mathcal{A}$$

induced by the tautological embedding $S \hookrightarrow S_P$ descends to an equivalence

$$\mathrm{Tr}_{\int_P \mathcal{A}} \left(\int_S \mathcal{A} \right) \simeq \int_{S_P} \mathcal{A}.$$

In particular, if $S = S_2 \sqcup S_2$ and if the right (resp. left) collar belong to S_1 (resp. S_2) then

$$\int_S \mathcal{A} \simeq \int_{S_1} \mathcal{A} \underset{\int_P \mathcal{A}}{\boxtimes} \int_{S_2} \mathcal{A}.$$

3.3. **The quantum structure sheaf.** A very important feature of factorization homology is that it produces for any surface not just category but a *pointed* category: since it is functorial with respect to embeddings, the unique inclusion $\emptyset \hookrightarrow S$ of the empty surface into a surface S induces a functor

$$D_S : \mathrm{Vect}_{\mathbb{C}} \longrightarrow \int_S \mathcal{A}.$$

Indeed, since factorization homology is monoidal, the image of $\emptyset$, the unit in Surf, is the unit in LFP which is $\mathrm{Vect}_{\mathbb{C}}$. Since D_S is linear it is completely determined by the image of $\mathbb{C} \in \mathrm{Vect}_{\mathbb{C}}$.

Definition 3.6. *Let $\mathcal{O}_S$ be the object $D_S(\mathbb{C}) \in \int_S \mathcal{A}$. We call $\mathcal{O}_S$ the distinguished object, or the quantum structure sheaf.*

3.4. **Interlude: quantization of shifted Poisson structures.** There is a general notion of n-shifted Poisson structure [PTVV13, CPT+17] which à priori requires the language of *derived* stack, although it makes sense for ordinary stacks for $n = 0, 1, 2$. Very informally, n-shifted Poisson structures on X are expected to arise as quasi-classical limits of deformations of $\mathrm{QCoh}(X)$ as an E_n-category. Note that an E_0-category means a pointed category, i.e. a pair $(\mathcal{C}, x \in \mathcal{C})$. In that case one should think of x as a deformation of the structure sheaf of X, so that $\mathrm{End}(x)$ is a deformation-quantization in the good ol' usual way of the Poisson algebra of global function on X.

On the one hand, given an n-shifted Poisson stack X, one can consider the Poisson sigma model $\mathrm{Map}(-, X)$. One of the main theorems of *loc. cit.* states that

[5] This structure is well-defined up to equivalence. To fix a particular "model" for it, in particular if we want a strict version, we'd need to make some choices. This can be done in exactly the same way as we did for the disc.

if M is an oriented k-dimensional manifold, then the mapping stack $\mathrm{Map}(M, X)$ inherits a canonical $n-k$ shifted Poisson structure. On the other hand, if $\mathcal{A}$ is an E_n-category quantizing that structure on X, then $\int_M \mathcal{A}$ is an E_{n-k}-category. The general expectation, which to the best of my knowledge hasn't be made precise yet, is that the latter should automatically be a quantization of the former.

In our setting, the classifying stack BG has a 2-shifted Poisson structure, of which $\mathrm{Rep}_q G$ is a deformation. The character stack of M can be identified with $\mathrm{Map}(M, BG)$ so that if $\dim M = 2$ we get a 0, i.e. ordinary, Poisson structure, which is the stacky version of the Atiyah–Bott–Goldman Poisson structure. All in all, if S is a surface then $\int_S \mathrm{Rep}_q G$ ought to produce a canonical quantization of $\underline{\mathrm{Ch}}(S)$ in the direction of that Poisson structure.

3.5. Extension of the Reshetikhin–Turaev functor to surfaces.

Definition 3.7. *let $\mathcal{A}$ be ribbon and let $\mathcal{T}_{\mathcal{A}}(S)$ be the category whose:*

- *objects are finite configuration of framed points on S, labelled by pairs (x, ϵ) of a compact object $x \in \mathcal{A}$ and $\epsilon \in \{+, -\}$*
- *morphisms are linear combinations of isotopy classes of framed oriented ribbon graphs embedded in $S \times I$ with strands labelled by compact objects of $\mathcal{A}$ and vertices labelled by morphisms as in Theorem 2.23.*

Let

$$\overrightarrow{\mathrm{Conf}}_n(S) := \{((z_1, v_1), \ldots, (z_n, v_n)), (z_i, v_i) \in US, i \neq j \Rightarrow z_i \neq z_j\}$$

be the framed configuration space of S, where US is the unit bundle of S. Again the map

$$\overrightarrow{\mathrm{Conf}}_n(S) \to \mathrm{Emb}\left(\bigsqcup_{i=1}^{n} D^2, S\right)$$

is an homotopy equivalence. Hence, every framed configuration C on S induces, by universal property of factorization homology, a functor

$$F_C : \mathcal{A}^{\boxtimes n} \longrightarrow \int_S \mathcal{A}.$$

Further, every labelling of that configuration by arbitrary objects $x_1, \ldots, x_n$ of $\mathcal{A}$ determines an object in $\int_S \mathcal{A}$, namely the image of $x_1 \boxtimes \ldots \boxtimes x_n$ through that functor. In the particular case where the configuration at hand is image of one of the one we fixed through an embedding $\iota : D^2 \hookrightarrow S$, then by construction it factors through the tensor product of $\mathcal{A}$:

$$F_C(x_1 \boxtimes \ldots \boxtimes x_n) = F_\iota(x_1 \otimes \ldots \otimes x_n).$$

Every framed braid in S, i.e. any path in the configuration space between two configurations C, C' such that the top and bottom of each strand are labelled by the same object induces an isomorphism

$$F_C(x_1 \boxtimes \ldots \boxtimes x_n) \cong F_{C'}(x_1 \boxtimes \ldots \boxtimes x_n)$$

in $\int_S \mathcal{A}$. In the category $\mathcal{T}_{\mathcal{A}}(S)$, we also have morphisms which are image in the embedding $D^2 \times I \hookrightarrow S \times I$ induced by any choice of ι of a morphism in $\mathcal{T}_{\mathcal{A}}$ (note that this include in particular cups and caps). It's clear that morphisms in $\mathcal{T}_{\mathcal{A}}(S)$ are in fact generated by those together with the paths in the configuration space that we considered above. Hence we've sketched a proof of the following:

Theorem 3.8. *The Reshetikhin–Turaev functor has a canonical extension*

$$\mathrm{RT}_{\mathcal{A}} : \mathcal{T}_{\mathcal{A}}(S) \longrightarrow \int_S \mathcal{A}.$$

4. The quantum skein category

4.1. **Quantum Schur–Weyl duality.** Let $V = \mathbb{C}^n \in \operatorname{Rep}_q \mathrm{GL}_N$ and let T_n be the algebra whose elements are formal linear combinations of framed oriented tangles, oriented downwards, with n inputs and n outputs where each point is labelled by a copy V_i of V. Since $\operatorname{Rep}_q \mathrm{GL}_N$ is ribbon, the Reshetikhin–Turaev functor induces an algebra map

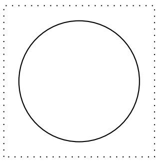
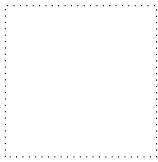
$$\mathrm{ev} : T_n \longrightarrow \operatorname{End}_{\operatorname{Rep}_q \mathrm{GL}_N}(V^{\otimes n}).$$

Let $\sigma \in S_n$ and let β_σ be the unique braid in T_n such that:

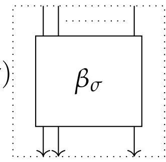
(a) there is a strand connecting the input labelled i with the output labelled $\sigma(i)$
(b) if $i < j$ then the strand starting at i always goes "above" the strand starting at j if and whenever they cross
(c) it has the smallest number of crossings among braids satisfying the two conditions above.

Theorem 4.1 ([Jim86]). *The map* ev *is surjective, its kernel is empty if $n \leq N$, and is otherwise generated by the following skein relations:*

(a) The "quantum dimension" relation

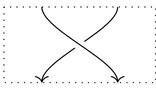
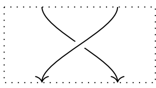
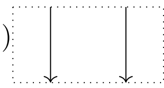
 $= \dfrac{q^{N/2} - q^{-N/2}}{q - q^{-1}} \cdot$  ,

(b) the q-antisymmetrizer relation

$\displaystyle\sum_{\sigma \in S_{N+1}} q^{-l(\sigma)} \epsilon(\sigma)$  $= 0$

where again l is the length and ϵ the sign

(c) the HOMFLY-Hecke relation

 $-$  $= (q - q^{-1})$ 

Remark 4.2. The reason for the factor $q^{1/N}$ in the definition of the R-matrix is that it makes the HOMFLY-Hecke relation independent of N.

4.2. **The skein category.** Let S be a compact oriented surface and let $\mathcal{T}(S)$ be the category whose objects are finite configurations of framed points on S labelled by $\{+, -\}$ and morphisms are formal linear combinations of framed oriented tangles in $S \times I$.

Definition 4.3. *The quantum skein category $\mathrm{Sk}_{q,N}(S)$ is the quotient of $\mathcal{T}(S)$ by the skein relations (a), (b) and (c).*

Theorem 4.4. *The generalized Reshetikhin–Turaev functor induces a fully faithful functor*

$$\mathrm{Sk}_{q,N}(S) \longrightarrow \int_S \operatorname{Rep}_q \mathrm{GL}_N .$$

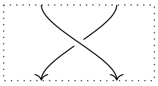
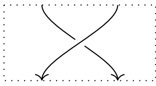
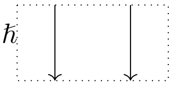
4.3. **The skein algebra and the Goldman bracket.**

Definition 4.5. *The* $(\mathrm{GL}_N -)$ *skein algebra of* S *is the algebra of endomorphism in* $\mathrm{Sk}_{q,N}(S)$ *of the empty configuration. Equivalently, its elements are formal linear combinations of framed oriented links in* $S \times I$ *modulo the skein relations.*

Theorem 4.6 ([Sik08, Tur91])**.** *The skein algebra is a quantization of the algebra of functions on the* GL_N *character variety of* S *in the direction of the Atiyah–Bott-Goldman Poisson bracket.*

Sketch of proof. We use the word "quantization" in a loose sense, the non-trivial part of that statement is that the skein algebra is in fact a *flat* deformation of the algebra of functions on the character variety. Here we just want to highlight how the HOMFLY-Hecke relation is related to the Goldman bracket.

Setting $q = \exp(\hbar)$ and expanding the last skein relation we get

 $-$  $= \hbar$  $+ O(\hbar^2).$

Just like for ordinary links in $\mathbb{R}^3$, a link L in $S \times I$ can be represented by a diagram drawn on S by choosing a representative of L in a sufficiently generic position, looking at its image on S through the projection π along I and labelling the intersection points as being either and under or an overcrossing. If L_1, L_2 are links in generic position, by definition one can go from the diagram of the product L_1L_2 to the diagram of L_2L_1 by flipping every crossing at the intersection of the diagrams of L_1 and L_2.

□

5. Explicit computation

From now on, we assume unless otherwise specified that $\mathcal{A}$ is braided, balanced, and rigid. To be on the safe side, we also assume it's abelian although this condition can often be weakened. The following is the first main result of those notes:

Theorem 5.1 ([BZBJ18a])**.** *Let* S *be a surface with non-empty boundary, together with the choice of an interval on one of the components of* ∂S*. To this data is canonically associated an algebra* $a_S \in \mathcal{A}$*, and there is an equivalence of*[6] $\mathcal{A}$*-modules*

$$\int_S \mathcal{A} \simeq a_s\text{-mod}_{\mathcal{A}}\,.$$

Under this identification, the functor

$$\mathcal{A} \longrightarrow \int_S \mathcal{A}$$

induced by the inclusion of a disc near the marked interval on the boundary of S *is given by the free module functor* $x \mapsto a_S \otimes x$*.*

The rest of this section is devoted to an explicit description of the algebras a_s, and to explain what happens for closed surfaces. More precisely, the algebra a_S itself is canonical (uniquely defined up to a unique isomorphism), and is non-canonically isomorphic to an explicitly described algebra, the isomorphism depending, roughly speaking, on a choice of free generators for $\pi_1(S)$.

The main motivation for that description is the following, which also connects this to the results in the first section:

[6] left or right, depending on the orientation of the marked interval

Theorem 5.2 (Ben-Zvi–Francis–Nadler)**.** *Let $\mathcal{A} = \operatorname{Rep} G$. Then there is an equivalence of categories*

$$\int_S \mathcal{A} \simeq \mathrm{QCoh}(\underline{\mathrm{Ch}}(S)).$$

5.1. **The key ingredient: the braided dual of $\mathcal{A}$.** In this section we introduce a certain canonical Hopf algebra object $\mathcal{O}_\mathcal{A}$ in a rigid braided tensor category $\mathcal{A}$. It should be thought of as an abstraction of the universal property of $\mathcal{O}(G)$ as well as of its Peter–Weyl description, and appears in the literature under various names: the braided dual or braided group of $\mathcal{A}$ [Maj95, Maj93], the reflection equation algebra [DKM03] and the loop algebra [Ale93]. We first make the following

Definition 5.3. *Let $\mathcal{A}$ be a braided tensor category with braiding β, and let a, b be algebra objects in $\mathcal{A}$ with multiplications μ_a, μ_b respectively. The braided tensor product $a\widetilde{\otimes}b$ is an algebra in $\mathcal{A}$ which as an object is $a \otimes b$ and with multiplication given by*

$$a \otimes a \otimes b \otimes b \xrightarrow{\mathrm{id}_a \otimes \beta_{a,b} \otimes \mathrm{id}_b} a \otimes b \otimes a \otimes b \xrightarrow{\mu_a \otimes \mu_b} a \otimes b.$$

If c, d are coalgebras, then their braided tensor product is defined similarly. A bialgebra in $\mathcal{A}$ is an object $h \in \mathcal{A}$ which is both an algebra and a coalgebra, and such that the multiplication is a map of coalgebras $h\widetilde{\otimes}h \to h$ (equivalently, the coproduct is a map of algebra $h \to h\widetilde{\otimes}h$).

Roughly speaking, we want to construct the universal coalgebra which coacts on any object in $\mathcal{A}$. More precisely, we want an object $\mathcal{O}_\mathcal{A}$ in $\mathcal{A}$, and maps

$$\Delta_x : x \longrightarrow \mathcal{O}_\mathcal{A} \otimes x$$

which are natural in x in the sense that for every morphism

$$\phi : x \longrightarrow y$$

the following diagram commutes

$$\begin{tikzcd} x \arrow[r, "\Delta_x"] \arrow[d, "\phi"] & \mathcal{O}_\mathcal{A} \otimes x \arrow[d, "\mathrm{id}_{\mathcal{O}_\mathcal{A}} \otimes \phi"] \\ y \arrow[r, "\Delta_y"] & \mathcal{O}_\mathcal{A} \otimes y \end{tikzcd}$$

and we want $\mathcal{O}_\mathcal{A}$ to be the universal such. In other words, we want $\mathcal{O}_\mathcal{A}$ to represent the functor

$$\begin{array}{rcl} \mathcal{A} & \longrightarrow & \mathrm{Vect}_\mathbb{C} \\ c & \longmapsto & \mathrm{Nat}(\mathrm{Id}_\mathcal{A}, \mathrm{Id}_\mathcal{A} \otimes c). \end{array}$$

Note that we are not requiring à priori $\mathcal{O}_\mathcal{A}$ to be a coalgebra, we actually get this for free from the naturality: the coproduct of $\mathcal{O}_\mathcal{A}$ will simply be given by $\Delta_{\mathcal{O}_\mathcal{A}}$. Assuming such an object exists, its square ${\mathcal{O}_\mathcal{A}}^{\otimes 2}$ likewise represents the functor

$$\begin{array}{rcl} \mathcal{A}^{\boxtimes 2} & \longrightarrow & \mathrm{Vect}_\mathbb{C} \\ c \boxtimes d & \longmapsto & \mathrm{Nat}(\mathrm{Id}_\mathcal{A}, \mathrm{Id}_\mathcal{A} \otimes c \otimes d). \end{array}$$

Moreover, we want a multiplication on $\mathcal{O}_\mathcal{A}$, inducing a tensor product on $\mathcal{O}_\mathcal{A}$-comodules extending that of $\mathcal{A}$ in the sense that the tensor product of x, y with their canonical comodule structure, ought to be $x \otimes y$ with its canonical comodule structure. One can use the braiding to turn $x \otimes y$ into a comodule over the braided tensor product coalgebra $\mathcal{O}_\mathcal{A}\widetilde{\otimes}\mathcal{O}_\mathcal{A}$, and we want a multiplication μ on $\mathcal{O}_\mathcal{A}$ which makes the following diagram commutes:

$$\begin{tikzcd} x\otimes y \arrow[r, "\widetilde{\Delta}_{x\otimes y}"] \arrow[d, "\mathrm{id}_{x\otimes y}"] & \mathcal{O}_{\mathcal{A}}\widetilde{\otimes}\mathcal{O}_{\mathcal{A}}\otimes x\otimes y \arrow[d, "\mu\otimes\mathrm{id}\, x\otimes y"] \\ x\otimes y \arrow[r, "\Delta_{x\otimes y}"] & \mathcal{O}_{\mathcal{A}}\otimes x\otimes y \end{tikzcd}$$

Since $\mathcal{A}$ is in LFP we can restrict to compact objects for the universal property of $\mathcal{O}_{\mathcal{A}}$, and since $\mathcal{A}$ is rigid, the map Δ_x can be dualized into a map

$$i_x : x\otimes x^* \longrightarrow \mathcal{O}_{\mathcal{A}}.$$

Naturality then translates to the fact that for every map $\phi : y \to x$ the following diagram should commutes

$$\begin{tikzcd} y\otimes x^* \arrow[r, "\phi^*"] \arrow[d, "\phi"] & y\otimes y^* \arrow[d, "i_y"] \\ x\otimes x^* \arrow[r, "i_x"] & \mathcal{O}_{\mathcal{A}} \end{tikzcd}$$

and $\mathcal{O}_{\mathcal{A}}$ should be universal for this property. The advantage of this description is that we have moved the "variables" all on one side, so that it characterizes maps out of $\mathcal{O}_{\mathcal{A}}$ which is typically what a colimit does. Indeed the above universal property is a particular case of what's called a coend:

$$\int^{\mathcal{A}_c} x\otimes x^*.$$

This object has a concrete description as the coequalizer of the diagram

$$\bigoplus_{\substack{\phi:y\to x\\ x,y\in I}} y\otimes x^* \underset{\phi\otimes\mathrm{id}}{\overset{\mathrm{id}\otimes\phi^*}{\rightrightarrows}} \bigoplus_{x\in I} x\otimes x^*$$

where I is any set of compact objects generating $\mathcal{A}$. This is well-defined since $\mathcal{A}$ is cocomplete, and we get the following "Peter-Weyl description" of $\mathcal{O}_{\mathcal{A}}$ and of its bialgebra structure:

Theorem 5.4. *Such an object exists, and is given by the "canonical coend"*

$$\mathcal{O}_{\mathcal{A}} := \int^{\mathcal{A}_c} x\otimes x^*.$$

The object $\mathcal{O}_{\mathcal{A}}$ has a canonical structure of an Hopf algebra in $\mathcal{A}$. The comultiplication is induced by the maps

$$x\otimes x^* \xrightarrow{\mathrm{id}_x\otimes\mathrm{coev}_x\,\mathrm{id}_x^*} (x\otimes x^*)\otimes(x\otimes x^*) \xrightarrow{i_x\otimes i_x} \mathcal{O}_{\mathcal{A}}\otimes\mathcal{O}_{\mathcal{A}}$$

and the multiplication by

$$(x\otimes x^*)\otimes(y\otimes y^*) \xrightarrow{\beta_{x^*,y^*}\beta_{x^*,y}} x\otimes y\otimes y^*\otimes x^* \cong (x\otimes y)\otimes(x\otimes y)^* \xrightarrow{i_{x\otimes y}} \mathcal{O}_{\mathcal{A}}.$$

The unit is given by the map $i_{\mathbf{1}_{\mathcal{A}}} : \mathbf{1}_{\mathcal{A}} \simeq \mathbf{1}_{\mathcal{A}}\otimes\mathbf{1}_{\mathcal{A}}^* \to \mathcal{O}_{\mathcal{A}}$ and the counit is induced by the evaluations $x\otimes x^* \to \mathbf{1}_{\mathcal{A}}$.

By construction every object is canonically an $\mathcal{O}_{\mathcal{A}}$-comodule, and if x is compact then the coaction is the composition

$$\Delta_x : x \xrightarrow{\mathrm{id}\otimes\mathrm{coev}_x} x\otimes x^*\otimes x \xrightarrow{i_x\otimes\mathrm{id}} x\otimes\mathcal{O}_{\mathcal{A}}.$$

This induces an automorphism L_x of the free module functor $\mathcal{O}_\mathcal{A} \otimes x$. Explicitly,

$$L_x : \mathcal{O}_\mathcal{A} \otimes x \xrightarrow{\mathrm{id}_{\mathcal{O}_\mathcal{A}} \otimes \Delta_x} \mathcal{O}_\mathcal{A} \otimes \mathcal{O}_\mathcal{A} \otimes x \xrightarrow{\mu_{\mathcal{O}_\mathcal{A}} \otimes \mathrm{id}_x} \mathcal{O}_\mathcal{A} \otimes x.$$

Theorem 5.5. *Let (H, r) be a co-quasitriangular Hopf algebra and let $\mathcal{A} = h$-comod. Then as an object in $\mathcal{A}$, $\mathcal{O}_\mathcal{A} = H$ equipped with the adjoint coaction, its coproduct is the one of H, but the multiplication is twisted by r.*

We will use the notation $\mathcal{O}_q(G) := \mathcal{O}_{\mathrm{Rep}_q G}$. As coalgebras we thus have $\mathcal{O}_q(G) = \mathrm{Fun}_q(G)$ but the multiplication is different. Indeed, observe that the operator $L_V^{(2)}$ of Definition 2.24 doesn't quite make sense internally to $\mathcal{A}$ since it involves the flip of 2-tensors in vector spaces. One needs to use the braiding instead, i.e. we set

$$\widetilde{L}_V^{(1)} = L_V \otimes \mathrm{id} \qquad\qquad \widetilde{L}_V^{(2)} = \beta_{V,V}^{-1}(L_V \otimes \mathrm{id})\beta_{V,V}.$$

Then, the multiplication in $\mathcal{O}_q(G)$ is characterized by the following relation:

$$\widetilde{L}_V^{(1)}\widetilde{L}_W^{(2)} = \widetilde{L}_W^{(2)}\widetilde{L}_V^{(1)}$$

Remark 5.6. Setting $q = \exp(\hbar)$ for a formal variable $\hbar$, we get an algebra $\mathcal{O}_\hbar(G)$ which is a flat deformation of $\mathcal{O}(G)$ in the direction of a certain Poisson structure on G discovered by Semenov-Tian-Shansky. This induces a Poisson structure on the categorical quotient

$$G//G \cong \mathrm{Ch}(S^1 \times I)$$

which coincides with the Atiyah–Bott–Goldman Poisson structure on the right hand side[7] . Let $\mathfrak{g}$ be the Lie algebra of G and let $\widehat{\mathcal{O}}_\hbar(\mathfrak{g}^*)$ be the subalgebra of $U(\mathfrak{g})[[\hbar]]$ (topologically) generated by $\hbar\mathfrak{g}$: this is a quantization of the linear Poisson structure on the formal neighborhood $\widehat{\mathfrak{g}^*}$ of 0 in $\mathfrak{g}^*$ induced by the Lie bracket of $\mathfrak{g}$. It follows from ? that one can construct an algebra map

$$\mathcal{O}_\hbar(G) \longrightarrow \widehat{\mathcal{O}}_\hbar(\mathfrak{g}^*)$$

which quantizes the so-called linearization map, a Poisson morphism

$$\widehat{\mathfrak{g}^*} \longrightarrow G.$$

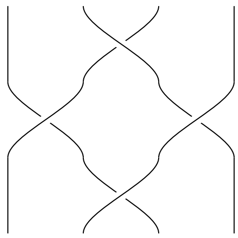
5.2. Factorization homology of the annulus.

Theorem 5.7 ([BZBJ18a]). *The algebra $a_{S^1 \times I}$ is canonically isomorphic to $\mathcal{O}_\mathcal{A}$.*

In particular, for every object x there is an action of the braid group of $S^1 \times I$ on the object $\mathcal{O}_\mathcal{A} \otimes x^{\otimes n}$ which maps the generator of $\pi_1(S^1 \times I)$ based at the "leftmost" point of any of our chosen configuration in the image of D^2, to the operator $L_x \otimes \mathrm{id}_x^{\otimes n-1}$.

[7]This might sound like a vacuous statement since the Atiyah–Bott–Goldman Poisson structure on $\mathrm{Ch}(S^1 \times I)$ turns out to be zero, as is easily seen from Goldman's formula, but this also induces the correct Poisson structure on the character stack.

5.3. **Quantum differential operators and the punctured torus.** Let c be the following isomorphism from $\mathcal{O}_{\mathcal{A}}^{\otimes 2}$ to itself:



Note that this makes sense since, using the coend formula, each copy of $\mathcal{O}_{\mathcal{A}}$ has two "slots" in which one can plug the braiding.

Definition 5.8. *The algebra of quantum differential operator $\mathcal{D}_{\mathcal{A}}$ is defined as follows: as an object it is $\mathcal{O}_{\mathcal{A}}^{\otimes 2}$, and the multiplication is defined by*

$$\mathcal{O}_{\mathcal{A}} \otimes \mathcal{O}_{\mathcal{A}} \otimes \mathcal{O}_{\mathcal{A}} \otimes \mathcal{O}_{\mathcal{A}} \xrightarrow{\mathrm{id} \otimes c \otimes \mathrm{id}} \mathcal{O}_{\mathcal{A}} \otimes \mathcal{O}_{\mathcal{A}} \otimes \mathcal{O}_{\mathcal{A}} \otimes \mathcal{O}_{\mathcal{A}} \xrightarrow{m \otimes m} \mathcal{O}_{\mathcal{A}} \otimes \mathcal{O}_{\mathcal{A}}$$

Theorem 5.9 ([BZBJ18a])**.** *There is an algebra isomorphism $a_{T^2 \setminus D^2} \simeq \mathcal{D}_{\mathcal{A}}$.*

Again, variants of this algebra (in the case where $\mathcal{A}$ is the category of representations of an Hopf algebra) appear in the literature under various names: for quantum groups it's closely related to the Heisenberg double [STS94] and coincides withe the handle algebra of [Ale93]. We set $\mathcal{D}_q(G) := \mathcal{D}_{\mathrm{Rep}_q G}$.

Remark 5.10. The name is justified by the following fact: the algebra $\mathcal{D}(G)$ of differential operators on G is isomorphic to the semi-direct product

$$\mathcal{D}(G) = \mathcal{O}(G) \rtimes U(\mathfrak{g})$$

where $U(\mathfrak{g})$ is identified with, say, left invariant differential operators on G. Setting $q = \exp(\hbar)$, the subalgebra

$$\mathcal{O}_{\hbar}(T^*G) = \mathcal{O}(G) \rtimes \mathcal{O}_{\hbar}(\mathfrak{g}^*) \subset \mathcal{D}(G)[[\hbar]]$$

is a quantization of the symplectic structure on the formal completion of $T^*G \simeq G \times \mathfrak{g}^*$ along the zero section. The map from Remark 5.6 extends to an algebra map

$$\mathcal{D}_{\hbar}(G) \longrightarrow \mathcal{D}(G)[[\hbar]]$$

which quantizes a (formal) Poisson morphism

$$T^*G \longrightarrow G \times G$$

where the Poisson structure on $G \times G$ was also introduced by Semenov–Tian–Shansky (the Heisenberg double). Hence this Poisson structure should be thought of as a multiplicative version of the cotangent bundle of G, and again it induces the Atiyah–Bott-Goldman Poisson structure on the character variety/stack of a punctured torus

$$\mathrm{Ch}(T^2 \setminus D^2) \cong (G \times G)/G.$$

5.4. **Punctured surfaces.**

Theorem 5.11 ([BZBJ18a])**.** *The algebra $a_{S_{g,n+1}}$ is isomorphic to $\mathcal{D}_{\mathcal{A}}^{\widetilde{\otimes} g} \widetilde{\otimes} \mathcal{O}_{\mathcal{A}}^{\widetilde{\otimes} n}$ where $\widetilde{\otimes}$ is the braided tensor product of algebras.*

Remark 5.12. This shows in particular that the GL_N skein algebra introduced in Section 4 is isomorphic to the subalgebra of invariants

$$\mathrm{Hom}_{\mathrm{Rep}_q G}(\mathbb{C}, \mathcal{D}_{\mathcal{A}}^{\widetilde{\otimes} g} \widetilde{\otimes} \mathcal{O}_{\mathcal{A}}^{\widetilde{\otimes} n}).$$

.

5.5. **Closed surfaces and quantum Hamiltonian reduction.** We know from abstract non-sense that $\int_{S^1\times I}\mathcal{A}$ should be monoidal. Since $\mathcal{O}_\mathcal{A}$ is a bialgebra, its category of modules is indeed monoidal, but it turns out this is *not* the correct monoidal structure[8].

Indeed, we also want the free module functor to be monoidal, i.e. the monoidal structure $\otimes_{cyl}$ we are looking for should satisfy

$$(\mathcal{O}_\mathcal{A}\otimes x)\otimes_{cyl}(\mathcal{O}_\mathcal{A}\otimes y)\cong\mathcal{O}_\mathcal{A}\otimes(x\otimes y).$$

So it seems what we want is something like the relative tensor product over $\mathcal{O}_\mathcal{A}$. In order to make sense of this, we make the following:

Definition 5.13. *Let $\mathcal{A}$ be a monoidal category, and a an algebra in $\mathcal{A}$ with multiplication μ. A commutative structure on a is a natural (in x) isomorphism*

$$\eta_x : a\otimes x \longrightarrow x\otimes a$$

such that

$$\eta_{x\otimes y}=\eta_y\circ\eta_x \qquad\qquad \mu\circ\eta_a=\mu$$

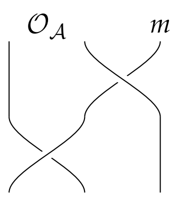
and such that $\eta_{\mathbf{1}_\mathcal{A}}=\mathrm{id}_{\mathbf{1}_\mathcal{A}}$.

Remark 5.14. If $\mathcal{B}$ is braided with braiding β, an algebra $a\in\mathcal{B}$ is called braided commutative if $\mu\circ\beta_{a,a}=\mu$. In particular, a commutative structure on a is the same as a lift of a into a braided commutative algebra in the Drinfeld center $\mathcal{B}=Z(\mathcal{A})$.

Proposition 5.15. *Let a be an algebra in $\mathcal{A}$ with a commutative structure η. Then a-$\mathrm{mod}_\mathcal{A}$ is monoidal, with tensor product given by the tensor product over a, using η to turn left modules into right ones. Formally, the tensor product of modules m,n with right action μ_m,μ_n respectively, is the coequalizer of*

$$m\otimes a\otimes n \underset{\mu_n\circ\eta_n}{\overset{\mu_m}{\rightrightarrows}} m\otimes n.$$

Proposition 5.16 (Majid)**.** *The algebra $\mathcal{O}_\mathcal{A}$ has a canonical commutative structure given by the "field goal transform"*

$\eta_m =$ 

.

This makes $\mathcal{O}_\mathcal{A}$-$\mathrm{mod}_\mathcal{A}$ into a monoidal category with tensor product $\otimes_{\mathcal{O}_\mathcal{A}}$.

Definition 5.17 ([Saf19])**.** *Let (a,μ) be an algebra in $\mathcal{A}$. A quantum moment map is an algebra map $\rho:\mathcal{O}_\mathcal{A}\longrightarrow a$ which is central, i.e. such that the following diagram commutes:*

[8]The category of $\mathcal{O}_\mathcal{A}$-module with that monoidal structure is, in fact, equivalent to the Drinfeld center of $\mathcal{A}$.

$$\begin{tikzcd}
\mathcal{O}_{\mathcal{A}} \otimes a \arrow[r, "\rho\otimes \mathrm{id}_a"] \arrow[dd, "\eta_a"] & a \otimes a \arrow[dr, "\mu"] & \\
& & a \\
a \otimes \mathcal{O}_{\mathcal{A}} \arrow[r, "\mathrm{id}_a \otimes \rho"] & a \otimes a \arrow[ur, "\mu"] &
\end{tikzcd}$$

Unpacking the definitions, one gets the following

Proposition 5.18. *There is an equivalence of categories between*

(a) algebras in the monoidal category $\mathcal{O}_{\mathcal{A}}\text{-mod}_{\mathcal{A}}$
(b) pairs of an algebra a in $\mathcal{A}$ and of a quantum moment map $\mathcal{O}_{\mathcal{A}} \to a$.

Definition 5.19. *Let a be an algebra with a quantum moment map ρ. A left a-module m is* strongly equivariant *if the* right *$\mathcal{O}_{\mathcal{A}}$-module structure on m obtained by pulling back along ρ and applying the field goal transform, is trivial.*

Remark 5.20. The counit $\epsilon : \mathcal{O}_{\mathcal{A}} \to \mathbf{1}_{\mathcal{A}}$ is trivially a quantum moment map, hence $\mathbf{1}_{\mathcal{A}}$ is an algebra in $\mathcal{O}_{\mathcal{A}}\text{-mod}_{\mathcal{A}}$. Then a strongly equivariant a-module is equivalently an $a - \mathbf{1}_{\mathcal{A}}$-bimodule in $\mathcal{O}_{\mathcal{A}}\text{-mod}_{\mathcal{A}}$.

Theorem 5.21 ([BZBJ18b])**.** *Let $S = S_{g,1}$. There exists a quantum moment map*

$$\rho : \mathcal{O}_{\mathcal{A}} \longrightarrow a_S$$

such that:

(a) The functor

$$\int_{S^1 \times I} \mathcal{A} \longrightarrow \int_S \mathcal{A}$$

induced by the inclusion of an annulus around the boundary component of S, is identified with the functor if induction along ρ, $m \mapsto a_s \otimes_{\mathcal{O}_{\mathcal{A}}} m$.

(b) There is an equivalence of categories between $\int_{S_g} \mathcal{A}$ and the category of strongly equivariant $a_{S_{g,1}}$-modules.

This quantum moment map can be described as follows: let γ be the loop based at the marked interval on $S_{g,1}$ going around the puncture. The formalism of factorization homology implies this induces an automorphism of the free $a_{S_{g,1}}$ module $a_{S_{g,1}} \otimes x$ for any compact x. By the free/forget adjunction this is the same as a map $x \to a_{S_{g,1}} \otimes x$ in $\mathcal{A}$, which can be dualized to a map

$$x \otimes x^* \longrightarrow a_{S_{g,1}}.$$

The claim is that, by the coend property, those assemble into an algebra map

$$\mathcal{O}_{\mathcal{A}} = \int^{\mathcal{A}_c} x \otimes x^* \longrightarrow a_{S_{g,1}}$$

which is the quantum moment map we were looking for.

Example 5.22. Remember that for $\mathcal{A} = \operatorname{Rep} G$, it is still true for a closed surface S that $\int_S \mathcal{A} = \operatorname{QCoh}(\operatorname{Ch}_S)$ is the category of $\mathcal{O}(\mathcal{R}_S)$-modules in $\operatorname{Rep} G$, but this relies in the fact that $\operatorname{Rep} G$ is symmetric. Still, in that case the theorem above recover our description of that category: the algebra $\mathcal{O}(G)$ is indeed tautologically commutative in $\operatorname{Rep} G$, and any algebra map is a moment map, so this is in particular the case for the algebra map

$$\rho : \mathcal{O}(G) \longrightarrow \mathcal{O}(\mathcal{R}_{S_{g,1}})$$

given by "taking the monodromy around the puncture" as in section 1.2. In that case, the theorem states that $\int_{S_g} \mathcal{A}$ is equivalent to the category of $\mathcal{O}(\mathcal{R}_{S_{g,1}})$-modules in $\operatorname{Rep} G$ which becomes trivial after pulling back through ρ. This is obviously the same as G-equivariant modules over $\mathcal{O}(\mathcal{R}_{S_{g,1}})/\rho(I)$ where I is the augmentation ideal[9], which is the same as $\mathcal{O}(\mathcal{R}_{S_g})$.

Remark 5.23. This is all motivated by the following construction: given a Poisson variety X with a compatible action of G, a moment map is a G-equivariant Poisson map $X \mapsto \mathfrak{g}^*$. This induces a Poisson structure on the quotient $\mu^{-1}(0)/G$. The fundamental example is a cotangent bundle $X = T^*Y$ of a (smooth) variety Y equipped with a G-action. This has a distinguished moment map ρ, and we have an identification

$$\rho^{-1}(0)/G = T^*(Y/G).$$

Now, thinking of $U(\mathfrak{g})$ as a quantization of $\mathfrak{g}^*$, and of the algebra $D(Y)$ of differential operators as a quantization of T^*Y, the Lie algebra map $\mathfrak{g} \to \operatorname{Der}(\mathcal{O}(Y))$ induced by the action of G extends to an algebra map $U(\mathfrak{g}) \to D(Y)$ which quantizes the above moment map. It turns out that $U(\mathfrak{g})$ has a canonical commutative structure as an algebra in $\operatorname{Rep} G$ (see [Saf19]), that this map is a quantum moment map is the sense above, and the category of strongly equivariant $D(Y)$-module is then identified with the category of D-modules on the quotient stack Y/G, a quantization of the cotangent stack $T^*(X/G)$. Thinking again, e.g., as the representation variety of a punctured torus, identified with $G \times G$, as a multiplicative version of T^*G, the map $G \times G \to G$ is a multiplicative moment map in the sense of [AMM98, AKSM02] (see also [Saf15]), and the character stack of the closed torus is obtained by Hamiltonian reduction. Hence, the character stack of T^2 is a multiplicative version of the cotangent stack $T^*(G/G)$. Accordingly, $\int_{T^2} \operatorname{Rep}_q G$ should be thought of as a category of "strongly equivariant quantum D-modules" on the stack G/G.

6. Proofs

6.1. **Reconstruction for module categories.** The goal of this section is to prove some kind of converse to Proposition ?. Let $\mathcal{A}$ be monoidal and $\mathcal{M}$ be a right $\mathcal{A}$-module. Let $m \in \mathcal{M}$ and consider the functor

$$\operatorname{act}_m : \mathcal{A} \longrightarrow \mathcal{M}$$

given by

$$\operatorname{act}_m(x) := m \triangleleft x.$$

This functor has a right adjoint, act_m^R which in general will not be a morphism in LFP since it may fail to be cocontinuous.

Definition 6.1. *Let $m, n \in \mathcal{M}$. We define the internal hom*

$$\underline{\operatorname{Hom}}_{\mathcal{M}}(m,n) := \operatorname{act}_m^R(n) \in \mathcal{A}.$$

[9]Here we mean a priori the quotient by the *left* ideal generated by $\rho(I)$, but here we're dealing with a commutative algebra so this ideal is two-sided in that case.

Remark 6.2. The name is justified by the following observation: let $\mathcal{A} = H\text{-mod}$ for an Hopf algebra H. For $V, W \in \mathcal{A}$, the vector space $\mathrm{Hom}_{\mathbb{C}}(V, W)$ (here we really mean all linear maps) is naturally an object in $\mathcal{A}$ via the adjoint action

$$(h \cdot f)(v) := h^{(2)} \triangleright f(S(h^{(1)}) \triangleright v)$$

where

$$\Delta(h) = \sum h^{(1)} \otimes h^{(2)}$$

is the coproduct and S the antipode. If V is finite dimensional this is just the module structure on $V^* \otimes W$. A straightforward modification of the tensor-hom adjunction gives a natural isomorphism

$$\mathrm{Hom}_H(V \otimes U, W) \cong \mathrm{Hom}_H(U, \mathrm{Hom}_{\mathbb{C}}(V, W)).$$

Hence, as an object in $\mathcal{A}$, $\mathrm{Hom}_{\mathbb{C}}(V, W)$ is the internal hom for the right action of $\mathcal{A}$ on itself by multiplication.

Playing with the adjunction we get composition maps, i.e. morphisms in $\mathcal{A}$ for all$m, n, p \in \mathcal{M}$

$$\underline{\mathrm{Hom}}_{\mathcal{M}}(m, n) \otimes \underline{\mathrm{Hom}}_{\mathcal{M}}(n, p) \longrightarrow \underline{\mathrm{Hom}}_{\mathcal{M}}(m, p)$$

which are associative in the obvious sense.

In particular, $\underline{\mathrm{Hom}}_{\mathcal{M}}(m, m)$ is an algebra object in $\mathcal{A}$ and $\underline{\mathrm{Hom}}_{\mathcal{M}}(m, n)$ is a left module over it.

The motivating example for this definition is as follows: let a be an algebra object in $\mathcal{A}$ and let $\mathcal{M} = a\text{-mod}_{\mathcal{A}}$. Regard a as a module over itself, then act_a is the free module functor, hence $\underline{\mathrm{Hom}}_{\mathcal{M}}(a, -)$ is just the forgetful functor $a\text{-mod}_{\mathcal{A}} \to \mathcal{A}$. In that case we claim the map

$$\underline{\mathrm{Hom}}_{\mathcal{M}}(a, a) \cong a$$

is in fact an isomorphism of algebra.

Theorem 6.3. *Let $m \in \mathcal{M}$ an object satisfying the following property:*

- *m is an $\mathcal{A}$-*generator*: the functor $\underline{\mathrm{Hom}}_{\mathcal{M}}(m, -)$ is conservative, i.e. for any morphism*

$$f : n \longrightarrow p$$

such that the induced map

$$\underline{\mathrm{Hom}}_{\mathcal{M}}(m, n) \to \underline{\mathrm{Hom}}_{\mathcal{M}}(m, p)$$

is an isomorphism in $\mathcal{A}$, f is an isomorphism in $\mathcal{M}$

- *m is $\mathcal{A}$-tiny (we also say $\mathcal{A}$-compact-projective): the functor $\underline{\mathrm{Hom}}_{\mathcal{M}}(m, -)$ is cocontinuous, i.e. is a morphism in* LFP*,*

then the canonical functor

$$\mathcal{M} \longrightarrow \underline{\mathrm{Hom}}_{\mathcal{M}}(m, m)\text{-mod}_{\mathcal{A}}$$

is an equivalence of $\mathcal{A}$-modules.

Remark 6.4. A faithful functor between abelian categories is automatically conservative, but this is not true in general.

This statement is a generalization of Gabriel's theorem which states that if $\mathcal{C}$ is an abelian category which is cocomplete and if $P \in \mathcal{C}$ is compact and projective and a generator (i.e. in the sense of Definition), then

$$\mathcal{C} \simeq \mathrm{End}(P)\text{-mod}.$$

6.2. **Reconstruction for monoidal categories.**

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