Swiss cheese, Poisson groups and topology

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Definition

- *A Poisson group is a (f.d., connected) affine algebraic group G over a field of char. 0 together with a Poisson structure on its commutative Hopf algebra of functions $\mathcal{O}(G)$, such that the coproduct (resp. antipode) is a Poisson (resp. anti-Poisson) map.*
- *A quantization of G is a (flat) Hopf algebra deformation of $\mathcal{O}(G)$ which also quantizes the underlying Poisson algebra.*

Theorem (Etingof–Kazhdan, Tamarkin)

For any choice of a Drinfeld associator[a] Φ, there exists a functorial quantization $G \mapsto \mathcal{O}_\Phi(G)$ of Poisson groups.

[a] An extremely complicated and non-canonical object that we need to pick only once and can then use as a black box.

- This talk:
  - I will review a somewhat folkloric TFT interpretation of their construction.
  - My goal is to emphasize that there is a lot more interesting structures and symmetries than the mere existence of these quantizations, made manifest by this perspective.
  - I will illustrate this in the seemingly trivial case where the Poisson structure is 0 (this is closely related to BF theory).

- Such Poisson structures are in 1 to 1 correspondence with Lie cobrackets

$$\delta : \mathfrak{g} \longrightarrow \mathfrak{g} \wedge \mathfrak{g}$$

on the Lie algebra $(\mathfrak{g}, \mu)$ of G, making it into a Lie bialgebra.

- This notion is self-dual:

$$\mathfrak{g}^\vee := (\mathfrak{g}^*, \delta^t, -\mu^t)$$

is again a Lie bialgebra.

- Pick a Poisson group $G^\vee$ with Lie bialgebra $\mathfrak{g}^\vee$.
- There exists a unique Lie algebra structure on $\mathfrak{g} := \mathfrak{g} \oplus \mathfrak{g}^\vee$ (the Drinfeld double) such that:
  - $\mathfrak{g}, \mathfrak{g}^\vee$ are Lie subalgebra
  - the tautological pairing on $\mathfrak{d}$ is ad-invariant.

- Let D be a group with Lie algebra $\mathfrak{d}$, and we'll pretend that $D = G \times G^\vee$ as varieties.

- The pairing on $\mathfrak{d}$ induces a canonical Poisson structure on character varieties (really stacks, aka moduli stacks of flat D-bundles)

$$\mathrm{Ch}_D(S) := \{\pi_1(S) \longrightarrow D\}/D$$

for any compact oriented (connected) surface S.

- This generalizes the Atiyah–Bott–Goldman Poisson structure on character varieties for reductive (or compact) groups.

- This is a 2d TFT, the AKSZ Poisson sigma model with target the classifying stack $BD = pt/D$.

- Compatible with smooth embeddings.

- Linearised version which attaches the categories of vector bundles/sheaves on D-character varieties to surfaces, and in particular $\operatorname{Rep} D$ to the disc.

- Objects of $\operatorname{Rep} D$ are point defects, the tensor product comes from fusion and the symmetric braiding from moving points around.

- $G, G^\vee$ defines boundary conditions: can mark intervals or circles on ∂S

$BG :=$ $\qquad BG^\vee :=$

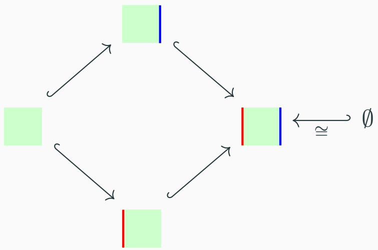
- Higher algebraic perspective (Safronov, (C)PTVV): BD is 2-shifted symplectic, and $BG, BG^\vee \to BD$ are Lagrangian.

- Those boundary conditions are transverse: the map

 $= G\backslash D/G^\vee \longrightarrow pt$

is an iso.

- Applying this to the diagram



- induces the commutative diagram of monoidal forgetful functors

$$
\begin{array}{ccccc}
 & & \mathrm{Rep}\ G^{\vee} & & \\
 & \nearrow & & \searrow & \\
\mathrm{Rep}\ D & & & & \mathrm{Vect} \\
 & \searrow & & \nearrow & \\
 & & \mathrm{Rep}\ G & &
\end{array}
$$

- After picking an associator Φ, everything has a canonical quantization.
- We get a balanced braided/framed E_2 category $\mathcal{A} = \mathrm{Rep}_\Phi\, D$. Crucially the underlying category is the same (only the braided structure is deformed).
- Since

$$BG = pt/G \cong D\backslash D/G \cong D\backslash G^\vee$$

  we have a monoidal equivalence

$$\mathrm{Rep}\, G \simeq \mathcal{O}(G^\vee)\text{-mod}_D$$

  where on the right the tensor product is over the commutative algebra $\mathcal{O}(G^\vee)$.
- With its original product it becomes a braided commutative/E_2-algebra in $\mathcal{A}$, and likewise for $\mathcal{O}(G)$.
- Taking modules over those, we get (right and left) Swiss–Cheese algebras, i.e. monoidal categories $\mathcal{B}$ and $\mathcal{C}$, together with braided functors

$$\mathcal{A}^{bop} \longrightarrow Z(\mathcal{B}) \qquad\qquad \mathcal{A} \longrightarrow Z(\mathcal{C})$$

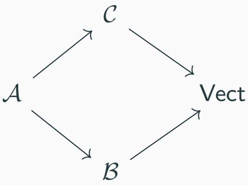
  where $Z(-)$ is the Drinfeld center and *bop* means braided opposite.

- Via the formalism of factorization homology/skein categories this leads to an oriented 2d TFT with two transverse boundary conditions: the unit

$$\mathrm{Vect} \longrightarrow \mathcal{B} \otimes_{\mathcal{A}} \mathcal{C}$$

is a monoidal equivalence.

- Again, we get a TFT which attaches pointed categories to compact oriented surfaces, possibly with red or blue marking on their boundary.
- Just like before, we get a commutative diagram of monoidal functors:



- The bottom right arrow, in particular, produces an Hopf algebra via Tannakian formalism: this is EK's quantization of G.

- But we get a lot more: the Swiss–cheese structures, and a manifest compatibility of the whole picture with $G \leftrightarrow G^\vee$.
- To spell it out: EK's quantization of G produces a quantization of $G^\vee$ as a byproduct and vice versa, and those are pairwise (non-trivially) isomorphic (Enriquez–Geer).
- The underlying functor of the composition

$$\mathrm{Rep}_\Phi D \longrightarrow \mathcal{B} \longrightarrow \mathrm{Vect}$$

  is still the forgetful functor, equipped with a highly non-trivial monoidal structure.
- This in turn is given by the action of an element $J_\Phi \in U(\mathfrak{d})^{\otimes 2}$ which can be written explicitly in terms of Φ.
- There is in fact a universal Feynman diagrammatic expression for J_Φ.
- The product on $\mathcal{O}_\Phi(G)$ is then

$$\mathcal{O}(G)^{\otimes 2} \xrightarrow{J_\Phi} \mathcal{O}(G)^{\otimes 2} \xrightarrow{m_0} \mathcal{O}(G)$$

  where m_0 is the original commutative multiplication.

- Let G be any (f.d. connected) affine algebraic group equipped with the 0 Poisson structure. Then:
  - $G^\vee = (\mathfrak{g}^*, +)$ with the linear KKS Poisson structure.
  - We have canonical isomorphism

$$\mathcal{O}_\Phi(G) \cong \mathcal{O}(G) \qquad\qquad \mathcal{O}_\Phi(G^\vee) \cong U(\mathfrak{g}).$$

- Here $U(\mathfrak{g})$ is seen as a quantization of $\mathfrak{g}^*$ via the PBW isomorphism

$$\mathrm{pbw} : S(\mathfrak{g}) \longrightarrow U(\mathfrak{g}).$$

- Recall

Theorem (Duflo, 77)

There exists an infinite order diff. operator u on $S(\mathfrak{g})$ such that

$$\mathrm{pbw} \circ u : S(\mathfrak{g})^G \longrightarrow U(\mathfrak{g})^G$$

is an algebra iso.

- Popular interpretation: $S(\mathfrak{g})^G$ is the Poisson center of $S(\mathfrak{g})$, $U(\mathfrak{g})^G$ is the center of $U(\mathfrak{g})$ and "quantization commutes with taking centers".

- Alekseev–Torossian (2006): proof of the Kashiwara–Vergne conjecture, a far reaching generalization of Duflo thm formulated in 76.

- Their proof uses associators and many algebraic ingredients looks like those in the EK formalism.

- Let $D = G \ltimes \mathfrak{g}^*$ where G acts via the adjoint action.

- Note that

  $$\mathfrak{g}^*/G = G \backslash D / G$$

  is the image of  via the classical TFT associated to this data.

- From the general formalism (this is a Lagrangian intersection) we recover that this is 1-shifted Poisson, hence that the underlying variety is Poisson commutative.

Theorem

The element $J_\Phi \in U(\mathfrak{d})^{\otimes 2}$ is isomorphic/gauge equivalent to one which acts trivially on $V^G \otimes W^G$ for all V, W in $\operatorname{Rep} D$.

- This has Duflo's theorem as an immediate corollary (take $V = W = S(\mathfrak{g})$).
- With lots and lots of hindsight, one recognizes this statement is essentially a reformulation of the KV conjecture.
- The proof uses the functoriality w.r.t.

  in order to show the statement for the $G^\vee$-invariants instead.

- Then we make crucial use of the $G \leftrightarrow G^\vee$ symmetry.

- Along the way we get two actions of the braid group B_n on $\mathcal{O}(G^n)$:
  - one by seeing $\mathcal{O}(G)$ as an object in $Z(\operatorname{Rep} G)$: I claim this is nothing but the mapping class group action on G^n seen as the G-representation variety of an n-punctured disc.
  - one by seeing $\mathcal{O}(G)$ as an object in $\operatorname{Rep}_\Phi D$: this one is closely related to the mapping class group action on flat $\mathfrak{g}$-connections on a punctured disc by isomonodromy.

Theorem

The Swiss–cheese structure, i.e. the braided functor

$$\operatorname{Rep}_\Phi D \longrightarrow Z(\operatorname{Rep} G),$$

intertwines these actions.

- This is a combinatorial version of the mapping class group equivariance of the Riemann–Hilbert correspondence.
- That solutions of the KV conjecture were related to these braid groups actions was observed in work of Bar-Natan–Dancso and Alekseev–Enriquez–Torossian: this is my attempt at an explanation.

Thanks for your attention !