Virtual knots and quantum groups

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Outline

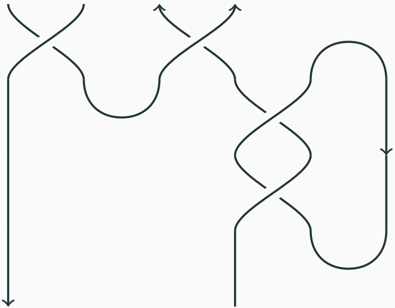
- I'll recall Shum–Reshetikhin–Turaev which gives a particular presentation by generators and relations of the category of tangles.
- This presentation is key to the way low-dimensional topology is related to deformation-quantization, in particular to quantum groups.
- However, knot theory "sees" only the category of representations of quantum groups, i.e. only quantizations of certain Poisson structures on the classifying space BG of algebraic groups.
- To see quantum groups themselves you need extra data, which in a way is like remembering the basepoint $pt \to BG$. This is naturally related to the quantum Yang-Baxter equation, and to Etingof–Kazhdan quantization of Poisson algebraic groups.
- Finally, I'll state a version of SRT theorem for *virtual* tangles which makes it clear that it captures precisely this extra data.

Ribbon categories

- Recall that a ribbon category is a monoidal category $(\mathcal{A}, \otimes)$ together with
  - a braiding
    $$\beta : V \otimes W \xrightarrow{\cong} W \otimes V$$
  - a dual V^* for each object V
  - a pivotal structure: a natural monoidal isomorphism $V \cong V^{**}$.
- Despite the fact that we nowadays use words like “braiding” and “ribbon”, I want to make a point that these axioms can be motivated purely representation theoretically and have nothing to do with topology *à priori*.

Shum and Reshetikhin–Turaev theorem

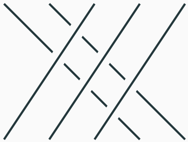
Let $\mathcal{T}$ be the monoidal category whose objects are finite words on $\{+,-\}$, whose tensor product is given by concatenation, and whose morphisms are given by framed oriented tangles



A morphism from $++--$ to $+-$

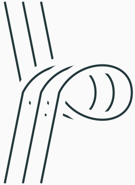
Shum and Reshetikhin–Turaev theorem

- Equip $\mathcal{T}$ with the braiding



- Equip $\mathcal{T}$ with a ribbon structure by declaring that $+$ and $-$ are mutually dual, that evaluation and coevaluation are given by

and by defining the pivotal structure to be



Shum and Reshetikhin–Turaev theorem

Theorem (Shum, Reshetikhin–Turaev)

$\mathcal{T}$ is the free ribbon category on one object.

- It means if $\mathcal{A}$ is a ribbon category, and $V \in \mathcal{A}$, then there is a unique functor $\mathcal{T} \to \mathcal{A}$ mapping $+$ to V.
- Gives easy-to-compute reps. of all the B_n's and links invariants.
- Gives a graphical way to handle braided and ribbon categories.

Quantum groups

- $\operatorname{Rep}_q G$: category of f.d. representation of the quantum group $U_q(\mathfrak{g})$ associated with a simply connected reductive complex algebraic group G, and a parameter $q \in \mathbb{C}^\times$.
- $\operatorname{Rep}_q G$ is a ribbon category. The braiding is induced by $\mathcal{R} \circ P$, where P is the swap of 2-tensors, and $\mathcal{R} \in U_q(\mathfrak{g})^{\otimes 2}$ satisfies some axioms implying in particular the quantum Yang–Baxter equation

$$\mathcal{R}^{1,2}\mathcal{R}^{1,3}\mathcal{R}^{2,3} = \mathcal{R}^{2,3}\mathcal{R}^{1,3}\mathcal{R}^{1,2}.$$

- Crucially, $\operatorname{Rep}_q G$ with its ribbon structure is *less* data (and in a way more canonical) than the data of $(U_q(\mathfrak{g}), \mathcal{R})$.
- This is nice, because that same category can be constructed in a more conceptual way, which in turn explains why it exists in the first place.
- To understand why $U_q(\mathfrak{g})$ exists, you need some additional data, which in a way is very much like the choice of a basepoint in a topological space.

The Drinfeld Kontsevich integral

- Let G be any f.d. algebraic group, and $t \in \mathfrak{g}^{\otimes 2}$ an invariant symmetric 2-tensor. E.g., if $\mathfrak{g} = \mathfrak{gl}_n$, then t is $\sum E_{ij} \otimes E_{ji}$ where E_{ij} is the elementary matrix.
- From this data, one can construct a system of differential equations on the configuration space of n points in $\mathbb{C}$, hence a family of representations of B_n. This is the so-called Knizhnik–Zamolodchikov equation and arises as a perturbation expansion of the WZW model in conformal field theory.
- Drinfeld showed the monodromy of this equation can be expressed algebraically in terms of a single object (a Drinfeld associator) and ultimately leads to a ribbon category $\mathrm{Rep}_{\mathrm{KZ}}\, G$ over the ring of formal power series $\mathbb{C}[[\hbar]]$.

The Drinfeld Kontsevich integral

- This uses SRT theorem the opposite way: you start with a geometric construction of braid groups actions on the trivial deformation Rep $G[[\hbar]]$, and then use their result to encode this structure using a finite amount of algebraic data.

Theorem (Kohno,Drinfeld)

If G is reductive and letting $q = \exp(\hbar)$, there is an equivalence of ribbon categories

$$\mathrm{Rep}_{\mathrm{KZ}}\, G \simeq \mathrm{Rep}_q\, G.$$

- This construction has a universal version where Rep G is replaced by some category of Feynman diagrams, which leads to a combinatorial construction of the Kontsevich integral of links.

Deformation-quantization

- The action of t turns $\operatorname{Rep} G$ into an infinitesimal braided monoidal category. This is the quasi-classical version of a braiding in the sense that in either $\operatorname{Rep}_q G$ or $\operatorname{Rep}_{\mathrm{KZ}} G$

$$\beta^2 = \mathrm{Id} + \hbar t + O(\hbar^2).$$

- This has a nice geometric interpretation: $\operatorname{Rep} G$ is identified with coherent sheaves on the "classifying stack" $BG = pt/G$ of G.
- Works of Pantev–Toën–Vaquie–Vezzosi explains that infinitesimal braided structures on QC of some geometric object X, are the same as so-called 2-shifted Poisson structures on X. Drinfeld associators gives a systematic way to quantize those.
- To recover G from BG one needs the basepoint $x : pt \to BG$ (morally $G = \pi_1(BG, x)$). This corresponds to the forgetful functor $\operatorname{Rep} G \to \mathrm{Vect}$.

Back to quantum groups

- So quantum knot invariants have something to do with quantizing some kind of Poisson structures on BG.
- On the other hand, quantum groups have something to do with quantizing multiplicative Poisson structures on G.
- Let $r \in \mathfrak{g}^{\otimes 2}$ be a solution of the classical Yang-Baxter equation

$$[r^{1,2}, r^{1,3}] + [r^{1,2}, r^{2,3}] + [r^{1,3}, r^{2,3}] = 0 \in \mathfrak{g}^{\otimes 3}$$

  such that $r^{1,2} + r^{2,1} = t$.
- Then r induces a(n ordinary) Poisson structure on G, with the property that the multiplication $G \times G \to G$ is a Poisson map. This is called a quasi-triangular Poisson algebraic group.
- Safronov showed this is equivalent to a (1-shifted) Poisson structure on the map $pt \to BG$.

Back to quantum groups

- If G reductive, there is a standard choice for r, and $\mathcal{R} = 1 + \hbar r + O(\hbar^2)$.
- It implies the Hopf dual $O_q(G)$ of $U_q(\mathfrak{g})$ is a quantization of the Poisson structure on G as an algebra, and encodes the product on G as a coalgebra.
- Etingof–Kazhdan showed that any quasi-triangular Poisson algebraic group (G, r) can be quantized to a quasi-triangular Hopf algebra $(U_\hbar(\mathfrak{g}), \mathcal{R})$.

Etingof-Kazhdan

- Tannakian formalism says Hopf algebras are the same as monoidal categories equipped with a fiber functor, i.e. a sufficiently nice, faithful, monoidal functor to Vect.
- The key ingredient of EK construction is a highly non-trivial monoidal structure on the forgetful functor

$$F : \mathrm{Rep}_{\mathrm{KZ}}\, G \longrightarrow \mathrm{Vect}$$

  i.e. a natural isomorphism

$$J_r : F(V) \otimes F(W) \longrightarrow F(V \otimes W)$$

  satisfying certain axioms.
- This shows $\mathrm{Rep}_{\mathrm{KZ}}\, G$ is equivalent to f.d. modules over a quasi-triangular Hopf algebra $(U_h(\mathfrak{g}), \mathcal{R})$.
- The triple $(\mathrm{Rep}_{\mathrm{KZ}}\, G, F, J_r)$ s a quantization of the Poisson structure on $pt \to BG$ induced by r.

Virtual tangles

- A natural question is: what is the topological meaning of this extra structure on $\mathrm{Rep}_{\mathrm{KZ}}\, G$?
- Virtual knots were introduced by Kauffmann: a knot diagram is a *planar* tetravalent graph with two types of vertices: overcrossing and undercrossings. So knot theory is the theory of those modulo the Reidemeister relations.
- Virtual knot theory is obtained by dropping the word "planar". The edges of the graph are allowed to intersect, and those are the virtual crossings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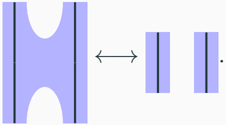
- This definition extends to braids and tangles in a straightforward way.

Virtual tangles

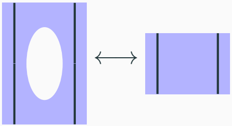
- The virtual crossings satisfy the defining relations of the symmetric group.
- If one maps the virtual crossing to the "swap" P, reps of B_n induced by a solution of the QYBE extend to reps of the virtual braid groups.
- Hence the virtual braid group is universal for the QYBE (as opposed to arbitrary braidings).
- Topological interpretation by choosing an abstract surface on which the diagram can be draw without self-intersection

Virtual tangles

- The choice of that surface is not unique, so you also need to mod out by two additional relations:
    - Tearing

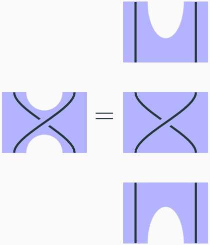


    - Puncturing



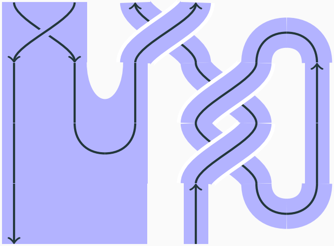
Virtual tangles

- By including the virtual crossing there is an obvious virtual version of the category of tangles.
- However, the topological picture makes it clear ordinary crossings can be chopped into more elementary pieces:



Virtual tangles

Define a category $v\mathcal{T}$ whose objects are finite sequences of words in $\{+, -\}$, and morphisms are framed oriented surfaces with non-empty boundary, intervals marked on the boundary and tangles drawn on them modulo Reidemeister relations, homeomorphisms, and tearing and puncturing.



A morphism from $[++][-][-]$ to $[+][-]$

A presentation of $v\mathcal{T}$

- $v\mathcal{T}$ is monoidal using concatenation of sequences, and symmetric monoidal using

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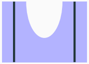
- There is a monoidal functor $\iota : \mathcal{T} \to v\mathcal{T}$ given on objects by $w \mapsto [w]$, and on morphisms by "embedding a tangle in a blue rectangle".
- The monoidal structure on ι and its inverse are given by

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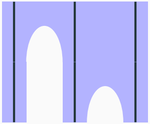
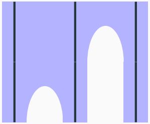
- Tearing and puncturing are precisely saying that those are mutually inverse.
- The fact ι is monoidal is just  $=$  .

A presentation of $v\mathcal{T}$

Theorem

The triple $(\mathcal{T}, v\mathcal{T}, \iota)$ is universal among triples of a ribbon category $\mathcal{A}$, a symmetric monoidal category $\mathcal{S}$, and a monoidal functor $F : \mathcal{A} \to \mathcal{S}$.

- This recovers the known fact that invariants coming from ribbon Hopf algebras extends to virtual links.
- This also applies to $(\mathrm{Rep}_{\mathrm{KZ}}\, G, \mathrm{Vect}, (F, J_r))$ as above.
- Of course Tannakian formalism says the latter can be "strictified" and reduced to the previous one. But if you care about EK it's nice not to do that and have a topological picture for J_r instead.
- This also encompass examples where the functor at hand is not necessarily faithful, or where the target is not Vect. Such examples arises in deformation-quantization.
- This whole picture was very much inspired by "Bar–Natan's dream" that there should be a diagrammatic version of EK coming from a version of the Drinfled–Kontsevich integral for virtual tangles.

Thanks for your attention !