# THE JOHNSON HOMOMORPHISMS

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## 1. $N$-series and associated graded of a f.g. group

Let $G$ be a fintely generated group, and for subgroups $A, B \subset G$ let $(A, B)$ be the subgroup generated by commutators $\{(a,b), a \in A, b \in B\}$.

**Definition 1.1.** *An $N$ series for $G$ is a sequence of subgroups*

$$G = \Phi^1 \supset \Phi^2 \supset \Phi^3 \dots$$

*such that* $(\Phi^m, \Phi^n) \subset \Phi^{m+n}$.

This implies at once that $\Phi^{m+1}$ is normal in $G$ (hence in $\Phi^m$) and that the quotient $\Phi^m/\Phi^{m+1}$ is abelian. The main example of an $N$-series is the lower central series defined by $\Gamma^1 = G$ and

$$\Gamma^{m+1} = (G, \Gamma^m).$$

An $N$-series is in particular central, so that $\Gamma^m \subset \Phi^m$, hence the quotient $G/\Phi^m$ is nilpotent. In particular, the subset of torsion element is a (normal) subgroup. The rationalization of $\Phi$ is

$$\Phi^m_{\mathbb{Q}} = \{x \in G, x^n \in \Phi^m \text{ for some } n\}.$$

It has the property that $G/\Phi^m_{\mathbb{Q}}$ is the quotient of $G/\Phi^m$ by its torsion subgroup.

**Definition 1.2.** *The associated graded w.r.t the series $\Phi$ is*

$$\mathrm{gr}^{\Phi} G = \bigoplus_{m \geq 1} \mathrm{gr}^m G$$

*where* $\mathrm{gr}^m_{\Phi} G := \Phi^m/\Phi^{m+1}$. *We set* $\mathrm{gr}\, G := \mathrm{gr}_{\Gamma} G$.

**Proposition 1.3.** *The commutator induces on $\mathrm{gr}_{\Phi} G$ the structure of a graded $\mathbb{Z}$-Lie algebra. The inclusion $\Gamma^m \subset \Phi^m$ induces a graded Lie algebra map*

$$\mathrm{gr}\, G \longrightarrow \mathrm{gr}_{\Phi} G.$$

*$\mathrm{gr}\, G$ is generated as a Lie algebra by $\mathrm{gr}_1 G$, i.e. the abelianization of $G$.*

*Sketch of proof.* Let $g \in G$, $x \in \Phi^m, y, y' \in \Phi^n, z \in \Phi^p$ and for $a, b \in G$ set $a^b = aba^{-1}$. Then:

- by defintion, $(x,y) \in \Phi^{m+n}$ and $(x,y)^g = (x,y) \mod \phi^{m+n+1}$.
- $(x, yy') = (x,y)(x,y')^y$ so the commutator descends to a bilinear map

$$\mathrm{gr}^m G \times \mathrm{gr}^n G \longrightarrow \mathrm{gr}^{m+n}$$

- the Hall-Witt identity

$$((x,y), z^x)\,((z,x), y^z)\,((y,z), x^y) = 1$$

  implies Jacobi.

□

**Warning 1.4.** *The map*

$$\mathrm{gr}\, G \longrightarrow \mathrm{gr}_{\Phi} G$$

*is neither injective or surjective in general, although it's obviously surjective in degree 1.*

**Theorem 1.5** (Magnus). *The associated graded ot the free group on n generators is the free Lie algebra on n generators. In particular, if $\pi = \pi_1(S_{g,1})$ and if $a_i, b_i$ is a symplectic basis of $H_1(S_{g,1})$, then $\operatorname{gr}\pi$ is the free Lie algebra on $a_i, b_i$.*

*The associated graded of $\pi_1(S_g)$ is the quotient of the former by the relation*

$$\sum [a_i, b_i] = 0.$$

## 2. Johnson homomorphisms

Let $A$ be a subgroup of $\operatorname{Aut}(G)$. Since $\Gamma^m$ is characteristic, there is a morphism

$$A \longrightarrow \operatorname{Aut}(G/\Gamma^m).$$

**Definition 2.1.** *The Johnson filtration is defined by:*

$$J^m := \ker(A \longrightarrow \operatorname{Aut}(G/\Gamma^{m+1})).$$

*The Torelli group of $A$ is $T_A := J^1$, a normal subgroup of $A$. The symmetry group of $T_A$ is $A_0 = A/T_A$.*

**Proposition 2.2** (Kaloujnine). *$J$ is an $N$ series on $T_A$.*

For a graded Lie algebra $\mathfrak{g}$, let $\operatorname{Der}^+(\mathfrak{g})$ be the Lie algebra of positive derivations

$$\operatorname{Der}^+(\mathfrak{g}) := \bigoplus_{m \geq 1} \operatorname{Der}^m(\mathfrak{g})$$

where $\operatorname{Der}^m(\mathfrak{g})$ is the space of derivations of $\mathfrak{g}$ which maps $\mathfrak{g}^n$ to $\mathfrak{g}^{n+m}$. Note this Lie algebra is itself graded. The following is an infinitesimal analog of the action of $T_A$ on $G$:

**Theorem 2.3** (Johnson, Papadima). *There is a well-defined,* injective *map of graded Lie algebra*

$$\tau : \operatorname{gr}_J(T_A) \hookrightarrow \operatorname{Der}^+(\operatorname{gr} G)$$

*called the Johson homomorphism, defined as follow: let $a \in J^m$, $x \in \Gamma^n$, then*

$$\bar{a} \cdot \bar{x} := \overline{a(x)x^{-1}}.$$

*Sketch of proof.* Let $a \in J^m$, $x \in \Gamma^n$. First we claim that

$$a(x) \equiv x \mod \Gamma^{m+n}.$$

For $n = 1$ this is the definition, for $n \geq 1$ this is proved by induction. Therefore,

$$a(x)x^{-1} \in \Gamma^{m+n}.$$

Composing with the quotient map we get a map

$$\Gamma^n \longrightarrow \operatorname{gr}^{m+n} G.$$

For $x, y$ in $\Gamma^n$, a direct computation shows that

$$a(xy)(xy)^{-1} \equiv (a(x)x^{-1})(a(y)y^{-1}) \mod \Gamma^{m+n+1}$$

hence this descends to an additive map

$$\operatorname{gr}^n G \longrightarrow \operatorname{gr}^{m+n} G.$$

By definition this map is the identity iff

$$\forall x \in \Gamma^n, a(x)x^{-1} \in \Gamma^{m+n+1}.$$

For $n = 1$ this says precisely that $a \in J^{m+1}$, and conversly every map in $J^{m+1}$ satisfies this for all $n$. Hence this map is injective. The fact that $a$ is a derivation, and that this map is a Lie algebra map follows from painful commutator computations. □

*Remark* 2.4. In a way the Johnson filtration is tailor made to make this map injective (it generally isn't for the lower central series).

The action of $A$ on $T_A$ by conjugation descends, essentially by construction, to an action of $A_0$ on $\mathrm{gr}_J(T_A)$ by graded Lie algebra automorphisms: for $a \in A$, $x \in T_A$,

$$\bar{a} \cdot \bar{x} := \overline{axa^{-1}}.$$

Likewise, it acts on $\mathrm{gr}\, G$ by

$$\bar{a} \cdot \bar{x} := \overline{a(x)},$$

hence on $\mathrm{Der}(\mathrm{gr}\, G)$ by the adjoint action:

$$\bar{a} \cdot d := \bar{a} \circ d \circ \bar{a}^{-1}$$

where $\circ$ is composition of endomorphisms.

**Proposition 2.5.** *The Johnson homomorphism is $A_0$-equivariant.*

## 3. Application to the actual Torelli groups

Let $S = S_{g,1}$ with a point $\star$ marked on the boundary and let $\pi = \pi_1(S, \star)$. Let $a_i, b_i$ be a symplectic basis of $H = H_1(S)$ and let $\omega = \sum a_i \wedge b_i$ the bivector associated with the symplectic form. Let $\zeta = [\partial S] \in \pi$ and recall the following classical

**Theorem 3.1** (Dehn)**.** *The natural map*

$$\mathrm{Mod}(S) \longrightarrow \mathrm{Aut}(\pi)$$

*is injective, and its image is the subgroup of automorphisms which fix $\zeta$.*

Identifying $\mathrm{Mod}(S)$ with its image, the associated Torelli group in the sense of the previous section is the usual Torelli group $T$, and the symmetry group $A_0$ is $\mathrm{Sp}(H)$.

*Remark* 3.2. One striking illustration of how useful it is to “linearize” $T$ in this way is that, as a consequence of a highly non-trivial theorem of Hain, we know that the Lie algebra $\mathbb{Q} \otimes \mathrm{gr}\, T$ is finitely presented (with an explicit presentation) for $g \geq 6$.

*Remark* 3.3. Since $\mathrm{Der}^+(\mathrm{gr}\, \pi)$ is torsion-free, and since $\mathrm{gr}_J\, T$ embeds into it, it is torsion free as well, which means that

$$J^m = J^m_{\mathbb{Q}}$$

and that

$$\Gamma^m_{\mathbb{Q}}(T) \subset J^m.$$

On the other hand, it is known that the abelianization of $T$ has torsion so the map

$$\mathrm{gr}\, T \longrightarrow \mathrm{gr}_J\, T$$

is already not injective in degree 1, i.e.

$$\Gamma^2 \subsetneq J^2.$$

Johnson’s theorem below implies however that $\Gamma^2_{\mathbb{Q}}(T) = J^2$. One might hope this is true for $m \geq 3$, but it’s not: Hain has shown that the kernel of

$$\Gamma^2_{\mathbb{Q}}(T)/\Gamma^3_{\mathbb{Q}}(T) \longrightarrow J^2/J^3$$

is isomorphic to $\mathbb{Z}$.

*Remark* 3.4. It’s well known that for finitely generated free groups, one has

$$\bigcap_{m \geq 1} \Gamma^m_{\mathbb{Q}} = \{1\}$$

i.e. those are residually torsion-free-nilpotent. It implies at once that in the Torelli group

$$\bigcap_{m \geq 1} J^m = \{1\}$$

Since $\Gamma^m_{\mathbb{Q}}(T) \subset J^m$, $T$ is itself residually-torsion-free nilpotent. This has cool consequences: it is in particular torsion free and residually nilpotent (but this is much stronger), residually $p$ for all $p$, residually finite and bi-orderable.

*Remark* 3.5. One can check that the Johnson homomorphism in that case actually lands in the Lie algebra of symplectic derivations, i.e. those mapping

$$\omega = \sum [a_i, b_i] \in \operatorname{gr}\pi$$

to 0. This is the infinitesimal counterpart of the fact that the mapping class group action on $\pi$ preserves $\zeta$.

Recall that $T$ is generated by bounding pairs, i.e. elements of the form $T_\alpha T_\beta^{-1}$ where $T_\alpha$ is the Dehn twist along $\alpha$ and $\alpha, \beta$ are disjoint non-separating simple closed curves such that $[\alpha] = [\beta] \neq 0$. Recall also that if $\gamma$ is a bounding simple closed loop then $T_\gamma \in T$.

**Theorem 3.6** (Johnson). *The images of the elements $T_\gamma$, $\gamma$ a bounding simple closed curve, in $H_1(T)$ are 2-torsion, hence their image through $\tau_1$ is 0. In fact they generate the kernel of the lift*

$$T \longrightarrow \operatorname{Der}^1(\operatorname{gr}\pi).$$

**Theorem 3.7** (Johnson). *Let $S'$ be the component of $S\backslash(\alpha \cup \beta)$ which doesn't contain the base point. Let $k$ be the genus of $S'$ and let $\{a_i, b_i\}$ be a symplectic basis of $H$ such that $[\alpha] = [a_{k+1}]$, and such that $\{a_1, b_1, \ldots, a_k, b_k, a_{k+1}\}$ is a basis of $H_1(S')$. Then*

$$\tau_1(f) = (\sum_{i=1}^{k} a_i \wedge b_i) \wedge a_{k+1}.$$

Since the Lie bracket is antisymmetric, and since there is no Jacobi relation in degree 2, the free $\mathbb{Z}$-module $\operatorname{gr}_2 \pi$ can be identified with $\wedge^2 H$. Therefore any linear map $H \to \wedge^2 H$ extends uniquely to a degree 1 derivation of $\operatorname{gr}\pi$, and using the symplectic form on $H$ to identify $H \cong H^*$, we get an $\operatorname{Sp}(H)$-equivariant embedding

$$\wedge^3 H \hookrightarrow H \otimes \wedge^2 H \cong H^* \otimes \wedge^2 H \cong \operatorname{Der}^1(\operatorname{gr}\pi).$$

**Theorem 3.8** (Johnson). *The first Johnson homomorphism*

$$\tau_1 : H_1(T) \longrightarrow \operatorname{Der}^1(\operatorname{gr}\pi)$$

*lands in $\wedge^3 H$, and induces an isomorphism*

$$H_1(T, \mathbb{Q}) \cong \wedge^3 H \otimes \mathbb{Q}.$$

*It descends to an isomorphism*

$$H_1(T_g, \mathbb{Q}) \cong (\wedge^3 H \otimes \mathbb{Q}) / \langle h \wedge \omega, h \in H \rangle$$

*where $T_g$ is the Torelli group of the closed surface $S_g$ and $\pi_g$ its fundamental group.*

## 4. Geometric Johnson homomorphisms

4.1. **Abel–Jacobi map.** Let $\mathbb{T}$ be the complex torus $\mathbb{C}^g/\mathbb{Z}^{2g}$. Since this is a $K(\mathbb{Z}^{2g}, 1)$, the abelinanization map $\pi \to \mathbb{Z}^{2g}$ determines a unique homotopy class of maps

$$S \longrightarrow \mathbb{T}.$$

The Abel–Jacobi map can be thought of as a way of picking representatives in that homotopy class in a way that interacts well with the action of $T$, by using complex structures on $S$. Fix once and for all a surface $\bar{S}$ obtained by gluing a disc to the boundary of $S$ and fix a marked point inside that disc with a unit tangent vector at it. By a complex structure on $S$ we'll mean a pair of a marked compact Riemann surface $C$ and of a diffeomorphism $h : C \xrightarrow{\sim} \bar{S}$ which preserves the basepoint and its tangent vector. Two complex structures $(C, h)$ and $(C', h')$ on $S$ are isotopic if $h^{-1} \circ h'$ is isotopic (rel. basepoint and tangent vector) to a holomorphic diffeomorphism.

Recall that the cotangent bundle of $C$ has a canonical holomorphic structure, called the canonical line bundle $K$, so that global sections $H^0(C, K)$ are identified with holomorphic one forms on $C$. It is well-known that this space is isomorphic as a real vector space to $H^1(\bar{S}, \mathbb{R})$.

If $\alpha$ is such a form, and $\gamma$ a path on $C$, define

$$\int_\gamma \alpha := \int_0^1 \gamma^*\alpha \in \mathbb{C}.$$

Integration of forms gives a non-degenerate pairing

$$H_1(C,\mathbb{C}) \times H^0(C,K) \longrightarrow \mathbb{C},$$

hence an embedding $H_1(C) \subset H^0(C,K)^*$.

**Definition 4.1.** *The Jacobian of $C$ is $J(C) := H^0(C,K)^*/H_1(C)$. The choice of a symplectic basis of $H_1$ induces an identification $J(C) \cong \mathbb{T}$. The Abel–Jacobi map*

$$j : C \longrightarrow J(C)$$

*is defined for $y \in C$ by picking a path $\gamma$ from the marked point to $y$, and mapping $y$ to*

$$\alpha \mapsto \int_\gamma \alpha.$$

Note that if $\gamma'$ is another path to $y$, then $\int_{\gamma^{-1}\gamma'} \alpha$ is 0 in $J(C)$, hence this is well-defined. It's also clear that this map induces the abelianization of $\pi$.

4.2. **Bundles over the Torelli space.** Let Teich be the Teichmüller space, whose points are sotopy classes of complex structures on $S$ as above. An important fact about

**Theorem 4.2.** *The space* Teich *is homeomorphic to $\mathbb{R}^{6g-3}$ and carries a free action of the Torelli group $T$.*

**Definition 4.3.** *The Torelli space $\mathcal{T}$ is the quotient* Teich $/T$.

A point in $\mathcal{T}$ is thus a pair (a diffeomorphism class of complex structures on $S$, a symplectic basis of $H_1(S)$). Note it carries a residual action of Sp by changing the basis. It follows from the theorem that $\mathcal{T}$ is a $K(T,1)$, hence we have:

$$H_*(\mathcal{T}) \cong H_*(T).$$

Let $\mathcal{T}^* = (\bar{S} \times \text{Teich})/T$ and let

$$p : \mathcal{T}^* \longrightarrow \mathcal{T}$$

be the projection. This is the homotopy quotient of $\bar{S}$ by $T$, i.e. it is the universal fiber bundle over $\mathcal{T}$ with fiber over any point identified with $\bar{S}$. Let now $\mathcal{J}^*$ be the trivial bundle

$$(\mathbb{T} \times \text{Teich})/T \longrightarrow \mathcal{T}$$

where $T$ acts on $\mathbb{T}$ trivially (hence this is indeed a trivial bundle). Fixing once and for all a symplectic basis of $H_1(S)$, for any $(C,h) \in \text{Teich}$, using $h$ this canonically fixes an identification $J(C) \simeq \mathbb{T}$. Therefore the Abel-Jacobi maps assemble into a map

$$\bar{S} \times \text{Teich} \longrightarrow \mathbb{T} \times \text{Teich}$$

which is clearly $T$-equivariant, since by construction the action of $T$ preserves any choice of a symplectic basis of $H_1(S)$. In other words, we get an Sp-equivariant bundle map

$$\mathcal{T}^* \longrightarrow \mathcal{J}^*.$$

Composing this map with the projection $\mathcal{J}^* \to \mathbb{T}$ we get an Sp-equivariant diagram

$$\mathcal{T} \longleftarrow \mathcal{T}^* \longrightarrow \mathbb{T}.$$

**Definition 4.4.** *Let $f : M \longrightarrow N$ be a smooth map between compact oriented manifolds of dimensions $m+k$ and $m$ respectively. The pull-back along $f$ in homology is the composition*

$$f^* : H_i(N) \xrightarrow{\text{Poincaré duality}} H^{m-i}(N) \xrightarrow{\text{pull-back along } f} H^{m-i}(M) \xrightarrow{\text{Poincaré duality}} H_{i+k}(M).$$

*Remark* 4.5. Let $X \subset N$ be a submanifold determining a class $[X] \in H_i(N)$. Then modulo transversality conditions, the pull-back of $[X]$ in $H_{i+k}(M)$ is represented by the preimage of $X$ in $M$.

**Definition 4.6.** *The ith geometric Johnson homomorphism is the composition*

$$\tau_i' : H_i(\mathcal{T}, \mathbb{Q}) \xrightarrow{p^*} H_{i+2}(\mathcal{T}^*, \mathbb{Q}) \longrightarrow H_{i+2}(\mathbb{T}, \mathbb{Q}) \cong \bigwedge^{i+2} H_{\mathbb{Q}}.$$

**Theorem 4.7** (Johnson, Hain, Church–Farb)**.** $\tau_1' = \tau_1$.

4.3. **Mapping tori and $\tau_1'$.** There is a fun way to compute $\tau_1 = \tau_1'$ this way. Let $\sigma \in H_i(\mathcal{T})$. Suppose we are given a map $\iota : B \longrightarrow \mathcal{T}$ and a class $x \in H_i(B)$ whose image in $H_i(\mathcal{T})$ is $\sigma$. Form the pull-back bundle

$$E = \{(x,t) \in \mathcal{T}^* \times B \mid p(x) = \iota(t)\}$$

Then we can compute $\tau_i'(\sigma)$ by running the above construction for the diagram

$$B \longleftarrow E \longrightarrow \mathbb{T}$$

instead. This is especially nice when $x$ is the fundamental class of $B$: remark 4.5 then says that $\tau_i(\sigma)$ is the image in $H_{i+2}(\mathbb{T})$ of the fundamental class of $E$. Often it will also be useful to replace the restriction to $E$ of the global Abel-Jacobi map by an homotopic one which is better adapted to some choice of coordinates on $E$. Then the basic idea is to look at certain degeneration of $\bar{S}$, i.e. roughly to run the construction by choosing a basepoint on the boundary of (some compactification of) $\mathcal{T}$ where things become easier to compute.

Let $f$ be a diffeomorphism of $S$ lifting an element of $T$ and fix a basepoint $x \in \mathcal{T}$. By definition $f$ induces a loop in $\mathcal{T}$, i.e. a map $\gamma : S^1 \longrightarrow \mathcal{T}$. In that case the pull-back bundle is nothing but the mapping torus $M_f$ of $f$. Since $f$ acts trivially on $H_1(S)$, there is a canonical decomposition

$$H_1(M_f) = H_1(S) \times H_1(S^1).$$

By construction the image of the fundamental class of $S^1$ in $H_1(\mathcal{T})$ is the class of $f$, and using this splitting its pull-back in $H_3(M_f)$ is the fundamental class. Therefore, $\tau_1(f)$ is the image of $[M_f]$ in $H_3(\mathbb{T}, \mathbb{Q})$.

This gives a cool geometric proof of theorem 3.7, or rather a proof that $\tau_1 = \tau_1'$ using that theorem, which we know sketch. As a warm-up let us compute $\tau_0'$. It follows from the discussion above that the image of the generator of $H_0(\mathcal{T})$ in $\bigwedge^2 H_{\mathbb{Q}}$ is simply $j_*[\bar{S}]$. One can pinch $\bar{S}$ into a wedge of $g$ tori and $j$ factors through the projection $\bar{S} \longrightarrow \bigvee T_i$. If $(a_i, b_i)$ is the chosen symplectic basis of $H_1(S)$, we identify $T_i$ with the 2-torus inside $\mathbb{T}$ determined by $a_i$ and $b_i$. This induces an inclusion

$$u : \bigvee T_i \hookrightarrow \mathbb{T}.$$

The compositon

$$\bar{S} \longrightarrow \bigvee T_i \longrightarrow \mathbb{T}$$

is homotopic to $j$. Hence $j_*[\bar{S}] = \sum u_*[T_i]$, which by construction equals $\omega = \sum a_i \wedge b_i$.

Similarly, to compute $\tau_1'$, the first step is to pick a nice lift

$$J : M_f \longrightarrow \mathbb{T}$$

i.e. a map

$$J : \bar{S} \times [0,1] \longrightarrow \mathbb{T}$$

whose restriction to $\bar{S} \times \{0\}$ is $j$ and whose restriction to $\bar{S} \times \{1\}$ is $j \circ f$. Since $f \in T$, the maps $j$ and $j \circ f$ are homotopic. Since $\mathbb{T}$ is an abelian group, it makes sense to consider the difference $j - j \circ f$ which is then homotopically trivial. Hence there is a unique map

$$\delta : \bar{S} \longrightarrow \mathbb{C}^g$$

which maps the chosen basepoint on $\bar{S}$ to 0 and which satisfies

$$j - j \circ f = \delta \mod \mathbb{Z}^{2g}.$$

We then set

$$J(x,t) := j(p) + t\delta(p).$$

Given a bounding pair $(\alpha, \beta)$ can pinch $\bar{S}$ to obtain a space $Y$ with a torus "in the middle" containing $\alpha, \beta$ and $k-1$ tori glued to a point on the left and $g-k-1$ tori glued to a point on the right. We choose a symplectic basis of $H_1(S)$ such that the image of $(a_i, b_i)$ in $H_1(Y)$ is a

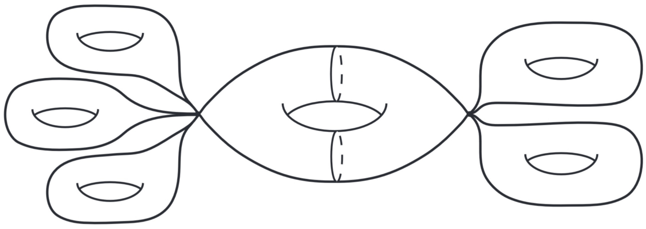

FIGURE 1. Degenerating $\bar{S}$ into a union of tori

basis of $H_1(T_i)$. There is a quotient map $\bar{S} \longrightarrow Y$ through which both $f$ and $j$ factor. Note that $f$ acts as the identity on every $T_i$ except for $i = k$.

We can then form the mapping torus $Z_f$ for $f$ on $Y$. It is the union of torus bundles

$$T_i \longrightarrow Z_i \longrightarrow S^1$$

meeting pairwise in at most a circle. We thus have that $\tau_1'(f)$ is $\sum J_*[Z_i]$.

For $i \neq k$ this bundle is trivial and $\delta$ is constant on $T_i$. Being careful in the choice of $j$ it follows that for $i < k$ the restriction of $J$ to $Z_i$ is given by

$$J(x,t) = j(x) + (0, \dots, t, \dots).$$

where $t$ is at the $k$th position. Thus, $J_*[Z_i]$ is $a_i \wedge b_i \wedge a_k$. When $k > i$ we have

$$J(x,t) = j(x)$$

so that $J_*[Z_i]$ is in the image of $H_3(T_i) \subset H_3(\mathbb{T})$. But $H_3(T_i)$ is trivial, so the image is 0 in that case. A similar argument shows that $J_*[Z_k] = 0$ as well.

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