What is. . . Higher algebra ?

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In a nutshell

- "Higher" means "higher dimensional", "higher homotopy groups" or perhaps "higher categorical"..
- For each $n \geq 0$ there is a notion of so-called E_n-algebra (aka algebra over the little n-discs operad) which for $n = 1$ essentially reduces to usual associative algebras, and which have many multiplications parametrized "locally constantly" by collapsing points in $\mathbb{R}^n$.
- The family $\{E_n\}_{n \geq 0}$ also have a nice inductive structure which gives a more concrete description of E_n-algebras.
- Those structures have a close connection with topological field theories, they also organize nicely a variety of interesting and concrete algebraic constructions.
- If one drops the locally constant property and/or replace $\mathbb{R}^n$ by an arbitrary (structured) manifold X one gets the notion of factorization algebra over X, which is expected to provide a formalization of observables in QFT's. In particular, factorization algebras on Riemann surfaces are close cousins of vertex algebras.

E_1-algebras are things with an associative multiplication

- An E_1-algebra has an $S^0 = \{\pm 1\}$ worth of multiplications, parametrized by collapsing points on the real line.
- Associativity corresponds to a path between the two different ways of collapsing 3 points. In vector spaces this gives ordinary algebras, but crucially this also makes sense in categories with "higher homotopies", i.e. invertible higher morphisms (more precisely so-called symmetric monoidal ∞-categories).
- Hence this also encompass up-to-homotopy versions of algebras where associativity becomes a structure rather than a property. Examples are monoidal categories (in Cat with natural isomorphisms) or A_∞-algebras (in chain complexes).
- More generally we can talk about E_1-algebras in any E_1/monoidal ∞-category: this is an instance of the so-called microcosm principle.

E_1-algebras are loop spaces

- Analogy: groups arises as π_1 of pointed spaces (X, x), or equivalently as π_0 of based loop spaces $\Omega_x X$ which are E_1-algebras.
- Every group arises this way by taking the classifying space.
- Similarly, if $(\mathcal{C}, x \in \mathcal{C})$ is a pointed ordinary or ∞-category, then its "based loop space"

  $$\Omega_x \mathcal{C} := \mathrm{End}_{\mathcal{C}}(x)$$

  is an E_1-algebra and

  $$\pi_1(\mathcal{C}, x) = \pi_0(\Omega_x \mathcal{C}) := \mathrm{End}_{\mathcal{C}}(x)/\sim$$

  is an ordinary algebra.
- Conversly, every (ordinary or up-to-homotopy) algebra arise this way from $(A\text{-mod}, A)$.

What is an E_n-algebra ?

- An E_n-algebra has an S^{n-1} worth of multiplications, parametrized by collapsing points in $\mathbb{R}^n$.
- If $n \geq 2$ the configuration space is connected, so all those multiplications are isomorphic. Hence to get something non-trivial we need ∞ categories to keep track of those isomorphisms.
- For familiar examples, in vector spaces E_n-algebras for $n \geq 2$ are just commutative algebras, while E_2-algebras in Cat are braided monoidal categories.

Commutative algebras

- Fun fact: if A is an (ordinary) algebra such that the multiplication $m : A^{\otimes 2} \to A$ is an algebra morphism, then A is commutative, i.e. commutative algebras are $Alg(Alg(vect))$.
- Applications:
  - π_2's are abelian groups
  - More generally, if $\mathcal{C}$ is an (ordinary) monoidal category, then $\mathrm{End}_{\mathcal{C}}(1_{\mathcal{C}})$ is a commutative algebra.
  - In particular, if A is any algebra, A-bimod is monoidal with unit A, so $\mathrm{End}_{A\text{-bimod}}(A)$ is commutative: this is the center of A.
  - The functor $Alg(vect) \to Cat$ given by $A \mapsto A$-mod is monoidal, hence send algebras to algebras, so if A is commutative, A-mod is a monoidal category.

E_2-algebras

- If we replace *Vect* by some symmetric monoidal ∞ category $\mathcal{S}$, then $Alg(Alg(\mathcal{S}))$ are exactly E_2-algebras. Crucially in this setting "being an algebra morphism" is a structure, not a property, hence so is being an E_2-algebra (think of monoidal functors). So:
  - Double loop spaces are E_2-algebras.
  - Braided monoidal categories are E_2-algebras in Cat.
  - By the microcosm principle we can thus consider E_2-algebras in those: those are braided commutative algebras.
  - If $\mathcal{C}$ is a monoidal category, then $\mathrm{End}_{\mathcal{C}}(1_{\mathcal{C}})$ is an E_2-algebra.
  - In particular, if A is any algebra in vect (or chain complexes), A-bimod is monoidal with unit A, so the derived $\mathrm{End}_{A\text{-bimod}}(A)$ is an E_2-algebra: this is Hochschild cohomology !
  - Similarly if $\mathcal{C}$ is a monoidal (i.e. E_1) category, its center $\mathrm{End}_{\mathcal{C}\text{-bimod}}(\mathcal{C})$ is E_2, i.e. braided: this is the Drinfeld center.
  - For the same reason as before, modules over an E_2-algebra is a monoidal category.

E_n-algebras

- The same pattern extends to all n: E_n-algebras are E_{n-1}-algebras in algebras.
- Hence ordinary modules over an E_n-algebras is an E_{n-1}-category.
- They also have a category of so-called operadic modules which is an E_n-category.
- More generally, playing with the identitification $E_n = E_{n-k}$ in E_k one gets a whole zoo of various notions of modules, corresponding to "codimension k defects".
- Every E_n-algebra have a center which an E_{n+1}-algebra.

A real-life example

- Monoidal categories also arise as H-mod for an Hopf algebra H.
- This is in fact a particular case: taken in an appropriate derived sense $H^\vee := \mathrm{End}_H(\mathbb{K})$ is an E_2-algebra and

$$H\text{-mod} \cong H^\vee\text{-mod}$$

  as monoidal categories.
- A lot of the theory of Hopf algebras can be naturally interpreted in this language, e.g.
  - Quasi-triangular Hopf algebras gives E_3-algebras.
  - "Codimension 1 modules" are comodule-algebras.
  - Operadic modules correspond to Hopf bimodules, which are equivalent to Yetter–Drinfeld modules.
  - The Drinfeld double of H is associated to the E_3-center of $H^\vee$ which is a variant of the Gerstenhaber–Schack complex.
  - ...