POISSON—LIE GROUPS. THE QUANTUM DUALITY PRINCIPLE AND THE TWISTED QUANTUM DOUBLE

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The quantum duality principle relates the quantum groups that arise on the quantization of Poisson—Lie dual groups and generalizes Fourier duality. Also considered are the theory of the Heisenberg double, which replaces the cotangent bundle for quantum groups, and its deformations (the twisted double).

The "theory of doubles" and its ramifications is a difficult part of the theory of quantum groups. The author does not know of any detailed exposition of this theory with all its motivations. This paper is an attempt to fill this gap partially. Mainly, we shall not consider the Drinfel'd double [1] but the related "Heisenberg" double, which for quantum groups plays the role of the cotangent bundle. Another important subject of the paper is the quantum duality principle (we describe it in detail in the Introduction). The length of the paper did not permit us to include the proofs, but I hope that the intuitive foundations of the constructions are described reasonably clearly. The main attention is devoted to the algebraic and geometric aspects of the matter; as regards analysis (topology of the algebras, completion, etc.), I have here allowed myself the liberty traditionally granted to physicists working with infinite-dimensional objects. At various times I have discussed the questions considered here with A. Yu. Alekseev, A. N. Kirillov, N. Yu. Reshetikhin, F. A. Smirnov, L. A. Takhtadzhyan, and L. D. Faddeev. I am sincerely grateful to all of them.

With deep sadness, I dedicate this paper to the memory of M. K. Polivanov.

1. INTRODUCTION

The theory of quantum groups is an attractive and increasingly popular generalization of the theory of Lie groups and algebras. Nontrivial examples of quantum groups are associated with quantum deformations of covering algebras of finite-dimensional semisimple and affine Lie algebras. The standard way of describing deformations is in terms of generators and relations (which generalize the classical Chevalley—Serret relations) [1,2]. The Faddeev—Reshetikhin—Takhtadzhyan dual approach [3,4] consists of the construction of the quantized covering algebras as deformations of affine rings of functions on Lie groups. The construction of the quantum deformation of the algebra Fun(G) was, of course, one of the first results of the theory of quantum groups and is a direct generalization of the Baxter commutation relations $RT_1T_2=T_2T_1R$. A nontrivial circumstance, first noted by Faddeev, Reshetikhin, and Takhtadzhyan, was that the dual algebra $\mathrm{Fun}_q(G)^* \simeq U_q(\mathfrak{G})$ can also be described as a deformation of a ring of functions on a Lie group (namely, on the dual group G^*, see below).*

A more general Faddeev—Reshetikhin—Takhtadzhyan construction is associated with the quantum duality principle, which we now formulate. We note first that in the semiclassical approximation the quantum deformations of function rings are associated with Poisson brackets on Lie groups. The class of Poisson structures associated with deformations of the algebras Fun(G) in the category of Hopf algebras is described by the following well-known axiom.

Definition (Drinfel'd [5]). *A Poisson bracket on a Lie group G defines on it the structure of a Poisson—Lie group if the multiplication*

$$m : G \times G \to G$$

is a Poisson mapping (see [6] for the general theory of Poisson manifolds).

A Poisson bracket that satisfies this condition is degenerate and vanishes at the identity of the group. Linearization of it at the identity defines the structure of a Lie algebra in the space $T_e^*G \simeq \mathfrak{G}^*$. The pair $(\mathfrak{G}, \mathfrak{G}^*)$ is called the *cotangent Lie bialgebra* of the group G. The Lie brackets on $\mathfrak{G}$ and $\mathfrak{G}^*$ satisfy the following **consistency condition**:

*In coded form, this circumstance was also noted in [1] ("equivalence of the category of *QFSH* algebras to the category of *QUE* algebras."

St Petersburg Branch of the V. A. Steklov Mathematics Institute, Russian Academy of Sciences. Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 93, No. 2, pp. 302-329, November, 1992. Original article submitted June 18, 1992.

 0040-5779/92/9302-1292$12.50 © 1993 Plenum Publishing Corporation

Let $\varphi : \mathfrak{G} \to \mathfrak{G} \wedge \mathfrak{G}$ be the mapping dual to the commutator

$$[,]_* : \mathfrak{G}^* \wedge \mathfrak{G}^* \to \mathfrak{G}^*.$$

Then φ is a 1-cocycle on $\mathfrak{G}$, i.e.,

$$\varphi([X,Y]) = [1 \otimes X + X \otimes 1, \varphi(Y)] - [1 \otimes Y + Y \otimes 1, \varphi(X)].$$

Let c_{ij}^k, f_c^{ab} be structure constants of the Lie algebras $\mathfrak{G}$, $\mathfrak{G}^*$ with respect to the basis $\{e_i\}$ in $\mathfrak{G}$ and the dual basis $\{e^i\}$ in $\mathfrak{G}^*$. Then the consistency condition means that

$$c_{ab}^s f_s^{jk} - c_{as}^j f_b^{sk} + c_{as}^k f_b^{sj} - c_{bs}^k f_a^{sj} + c_{bs}^j f_a^{sk} = 0.$$

This condition is symmetric with respect to c, f; thus, if $(\mathfrak{G}, \mathfrak{G}^*)$ is a Lie bialgebra, then $(\mathfrak{G}^*, \mathfrak{G})$ is also a Lie bialgebra. Let G^* be a Lie group with Lie algebra $\mathfrak{G}^*$. Since the correspondence between Lie bialgebras and Poisson—Lie groups is functorial [5], G^* is also a Poisson—Lie group.

Going over to the quantization, we can (provided the hindrances are equal to zero, see [1]) construct from given Poisson algebras Fun(G), Fun(G^*) two Hopf algebras $\mathrm{Fun}_q(G)$ and $\mathrm{Fun}_q(G^*)$. The quantum duality principle is that these algebras are duals of each other as Hopf algebras.

More precisely: *Let h be the deformation parameter. There is defined a nondegenerate bilinear pairing*

$$\mathrm{Fun}_q(G) \otimes \mathrm{Fun}_q(G^*) \to \mathbb{C}[[h]],$$

which reduces the algebras $\mathrm{Fun}_q(G)$, $\mathrm{Fun}_q(G^*)$ *to duality as Hopf algebras.* From this, in particular,

$$\mathrm{Fun}_q(G^*) \simeq U_q(\mathfrak{G}).$$

By duality, we also have

$$\mathrm{Fun}_q(G) \simeq U_q(\mathfrak{G}^*).$$

As a simple example, we consider a trivial Lie bialgebra. Let $\mathfrak{G}$ be an arbitrary Lie algebra, and $\mathfrak{G}^*$ be its dual space equipped with null Lie bracket. In this case, the Poisson bracket on G is null. The dual group is the additive group of the space $\mathfrak{G}^*$ equipped with the Lie—Poisson bracket [6,7] of the algebra $\mathfrak{G}$. On quantization, the algebra Fun(G) does not change in this case (since the Poisson bracket on G that defines the germ of the deformation is null). The deformation of the algebra $\mathrm{Fun}(\mathfrak{G}^*)$ can be identified with the universal covering algebra $U(\mathfrak{G})$ [7], where $\psi \in \mathrm{Fun}(\mathfrak{G}^*)$ is regarded as the symbol of the differential operator on the group G. The formula for the pairing $\mathrm{Fun}(G) \otimes \mathrm{Fun}_q(\mathfrak{G}^*) \to \mathbb{C}[[h]]$ has the form

$$\langle \varphi, \psi \rangle = \int_{G \times \mathfrak{G}^*} \varphi(x)\psi(p) \exp\left\{ \frac{i}{h} \langle p, \log x \rangle \right\} dx\, dp. \tag{1.1}$$

The measure $dx\,dp$ on $G \times \mathfrak{G}^* \simeq T^*G$ is, of course, the Liouville measure on the cotangent bundle T^*G. The appearance of T^*G in this context is not fortuitous. The pairing (1.1) also canonically generates the action

$$\mathrm{Fun}_q(\mathfrak{G}^*) \otimes \mathrm{Fun}\, G \to \mathrm{Fun}\, G : \quad \psi \otimes \varphi \mapsto \langle \Delta\varphi, \mathrm{id} \otimes \psi \rangle,$$

the usual action of $U(\mathfrak{G})$ on Fun(G) by left-invariant differentiations. Let $\mathcal{A}$ be the associative algebra generated by $U(\mathfrak{G})$ and Fun(G) regarded as operators of differentiation and multiplication in Fun(G). Then $\mathcal{A}$ is a quantization of the Poisson algebra of functions on T^*G with canonical Poisson bracket; the algebra $\mathcal{A}$ arises together with its irreducible (Schrödinger) representation.

Thus, even in the case of trivial Lie algebras the quantum duality principle is nontrivial — it includes, for example, the theory of Fourier transformation (and, in the Abelian case, the theory of Pontryagin duality). In the general case, the quantum duality principle can also be regarded as a generalization of Fourier transformation theory (cf. Sec. 4).

For semisimple Lie algebras, quantum deformations of the Poisson algebras Fun(G) and Fun(G^*) can be readily constructed if the corresponding quantum R matrices are known; this also makes it possible to verify the quantum duality principle. The need for *a priori* knowledge of the quantum R matrices can be regarded as a certain shortcoming of the Faddeev—Reshetikhin—Takhtadzhyan method. However, this shortcoming is offset not only by the geometrical transparency of the construction. More important is the fact that the Faddeev—Reshetikhin—Takhtadzhyan construction and its generalizations make it possible to include the algebras $\mathrm{Fun}_q(G)$ and $\mathrm{Fun}_q(G^*)$ in the complete association of related algebras. In the first place, we have here the quantum analog of the algebra Fun(T^*G) — the "Heisenberg double" of the quantum group $\mathrm{Fun}_q(D_+)$.

For ordinary Lie groups, $D_+ = T^*G$, and $\mathrm{Fun}_q(D_+)$ is the Heisenberg algebra generated by the operators of differentiation and multiplication on the group G. We recall that the connection between the canonical Poisson bracket on T^*G and the Lie—Poisson bracket on $\mathfrak{G}^*$ consists mainly of the usual geometrical construction of irreducible representations of the group G ("orbit method"). The connections between the algebras $\mathrm{Fun}_q(G)$, $\mathrm{Fun}_q(G^*)$, and $\mathrm{Fun}_q(D_+)$ are the exact analog of the orbit method. Because of lack of space, we shall not present here in detail this aspect of the theory. I hope to describe the "quantum orbit method" in a separate paper.

In many interesting cases, the algebra $\mathrm{Fun}_q(D_+)$ admits nontrivial deformation ("twisted double"). In particular, this makes it possible to construct nontrivial deformations $U_q(\mathfrak{G})^{\otimes N}$ that are the exact analog of the central extension of the loop algebra $L\mathfrak{G}$ [8].

The logic of the construction for general semisimple or affine algebras is as follows:

1. We use the Drinfel'd—Jimbo construction to construct the deformation in terms of generators and relations.

2. The theory of the universal R matrix [9,10,11] and the theory of representations of $U_q(\mathfrak{G})$ give expressions for the quantum R matrices and for the generators of the rings $\mathrm{Fun}_q(G)$ and $\mathrm{Fun}_q(G^*)$ in terms of the Drinfel'd—Jimbo generators.

3. After this, we can also describe all the related algebras (double, twisted double, and their subalgebras), which do not admit a simple description by means of "root" generators. In this paper, we shall not consider questions relating to the construction of universal R matrices and R matrices in specific representations, and we assume (in the spirit of [3]) that the necessary R matrices are known in advance.

2. SEMICLASSICAL DUALITY THEORY

Let $(\mathfrak{G}, \mathfrak{G}^*)$ be a Lie bialgebra, $\mathfrak{d} = \mathfrak{G}\dot{+}\mathfrak{G}^*$. On $\mathfrak{d}$ there exists [5] a (unique) Lie algebra structure such that:

1) $\mathfrak{G}, \mathfrak{G}^* \subset \mathfrak{d}$ *are Lie subalgebras,*

2) *the canonical bilinear form on* $\mathfrak{d}$,

$$\langle (X, f), (X', f')\rangle = f(X') + f'(X) \tag{2.1}$$

is ad $\mathfrak{d}$ *invariant.*

Let $P_{\mathfrak{G}}$, $P_{\mathfrak{G}^*}$ be the projectors onto $\mathfrak{G}, \mathfrak{G}^* \subset \mathfrak{d}$ in the decomposition $\mathfrak{d} = \mathfrak{G}\dot{+}\mathfrak{G}^*$. The operator

$$r_{\mathfrak{d}} = P_{\mathfrak{G}} - P_{\mathfrak{G}^*} \tag{2.2}$$

is skew symmetric and can be identified with the element $\Lambda^2\mathfrak{d}$. The formula

$$[X, Y]_* = \frac{1}{2}([r_{\mathfrak{d}}X, Y] + [X, r_{\mathfrak{d}}Y]) \tag{2.3}$$

defines in the space $\mathfrak{d}^* \simeq \mathfrak{d}$ a Lie bracket that transforms $(\mathfrak{d}, \mathfrak{d}^*)$ into a Lie bialgebra. It is readily seen that in fact

$$\mathfrak{d}^* \simeq \mathfrak{G} \oplus \mathfrak{G}^* \tag{2.4}$$

(direct sum of Lie algebras).

The bialgebra $(\mathfrak{d}, \mathfrak{d}^*)$ is called the double $(\mathfrak{G}, \mathfrak{G}^*)$. It is clear that the bialgebras $(\mathfrak{G}, \mathfrak{G}^*)$ and $(\mathfrak{G}^*, \mathfrak{G})$ have a common double. Let D be a Lie group with Lie algebra $\mathfrak{d}$. Many important Poisson structures are defined on D. The simplest of them are described as follows. Let φ, $\psi \in C^\infty(D)$. Let X_φ, X'_φ be the left and right gradients of φ, defined by

$$\langle X_\varphi(x), \xi\rangle = \left(\frac{d}{dt}\right)_{t=0} \varphi(e^{t\xi}x), \quad \langle X'_\varphi(x), \xi\rangle = \left(\frac{d}{dt}\right)_{t=0} \varphi(xe^{t\xi}), \quad \xi \in \mathfrak{d}. \tag{2.5}$$

We set

$$\{\varphi, \psi\}_\pm = \frac{1}{2}\langle r_{\mathfrak{d}}X_\varphi, X_\psi\rangle \pm \frac{1}{2}\langle r_{\mathfrak{d}}X'_\varphi, X'_\psi\rangle. \tag{2.6$_\pm$}$$

The bracket $\{,\}$ transforms the group D into a Poisson—Lie group, and its tangent Lie bialgebra is precisely $(\mathfrak{d}, \mathfrak{d}^*)$. The bracket $\{,\}_+$ is nondegenerate and defines on D a symplectic structure. If $(\mathfrak{G}, \mathfrak{G}^*)$ is a trivial Lie bialgebra, then $D = G \ltimes \mathfrak{G}^*$ is a semidirect product of the group G and the additive group $\mathfrak{G}^*$. Thus, in this case D can be identified with the cotangent bundle T^*G. It is readily seen that the bracket $\{,\}_+$ is identical to the canonical bracket on T^*G. The bracket $\{,\}$ is strongly degenerate — it is the direct product of the Lie—Poisson bracket on $\mathfrak{G}^*$ and the trivial bracket on G.

In the general case, the bracket $\{,\}_+$ is also the analog of the canonical bracket on the cotangent bundle. To describe its connection with the Poisson brackets on G and G^*, we first recall some simple facts on Poisson reduction (which generalizes ordinary Hamiltonian reduction) [12,13].

Let M be a Poisson manifold. The group action $H\times M\to M$ is said to be *admissible* if the space $\mathrm{Fun}(M)^H$ of H-invariant functions is a Lie subalgebra in the Poisson algebra $\mathrm{Fun}(M)$. Generally speaking, admissible actions do not necessarily preserve the Poisson bracket on M. If $H\times M\to M$ is an admissible action and the factor space M/H is a smooth manifold, then $\mathrm{Fun}(M/H)$ and $\mathrm{Fun}(M)^H$ can be identified. Thus, on M/H there is defined a Poisson structure such that the canonical projection $\pi: M\to M/H$ is a Poisson mapping. One says that M/H is obtained from M by a *Poisson reduction*. Even if M is symplectic, the bracket on M/H is, in general, degenerate. The nontrivial part of the reduction is the description of its symplectic sheets. This problem is solved by means of the concept of dual pairs [6].

We shall say that admissible actions of the groups H and H' on M are *dual* to each other if the subalgebras $\mathrm{Fun}(M)^H$ and $\mathrm{Fun}(M)^{H'}$ are the centralizers of each other in the Poisson algebra $\mathrm{Fun}(M)$. We assume that the manifold M is symplectic and that H and H' are dual groups of transformations of M. In this case, the symplectic sheets in M/H are connected components of the sets $\pi(\pi'^{-1}(x))$, $x\in M/H'$; similarly, the symplectic sheets in M/H' are connected components of the sets $\pi'(\pi^{-1}(y))$ and $y\in M/H$ [6].

THEOREM 2.1 [12]. *Let G and G^* be dual Poisson—Lie groups, and D_+ be their double with the Poisson bracket* $(2.6)_+$. 1. *The actions of the groups G and G^* on D_+ by right and left shifts are admissible.* 2. *The actions of the group G (G^*) by right and left shifts form dual pairs.*

An open everywhere dense subset of elements in D containing the neighborhood of the identity admits the unique factorization

$$x = gg^* = \tilde{g}^*\tilde{g}, \quad g, \tilde{g} \in G, \quad g^*, \tilde{g}^* \in G^*.$$

Thus, the group G can be identified with an open everywhere dense subset in D/G^*, $G^*\backslash D$, and the group G^* with an open everywhere dense subset in D/G, $G\backslash D$.

THEOREM 2.2 [12]. 1. *$G\subset D/G^*$ is a Poisson submanifold; the induced Poisson structure on G is identical to the original one.* 2. *Similarly, $G^*\subset D/G$ is a Poisson submanifold, and the induced structure on G^* is identical to the original one.*

For trivial Lie bialgebras, $D=T^*G$, $D/G=G\backslash D=\mathfrak{G}^*$; then Theorem 2.2 is transformed into the well-known result on the connection between the canonical Poisson bracket on T^*G and the Lie—Poisson bracket on $\mathfrak{G}^*$, and the dual pair of Theorem 2.1 becomes a theorem stating the commutation of the right and left moments; this theorem is the basis of the classical "orbit method" [14].

The main thing of interest for us is the case of the so-called *factorizable Lie bialgebras* [15]. Let $\mathfrak{G}$ be a semisimple Lie algebra equipped with a fixed nondegenerate invariant scalar product. We identify by means of it the spaces $\mathfrak{G}^*$ and $\mathfrak{G}$. The structure of the bialgebras on $\mathfrak{G}$ is given by the cobracket

$$\varphi(X) = -\frac{1}{2}[X\otimes 1 + 1\otimes X, r], \tag{2.7}$$

where $r \in \Lambda^2\mathfrak{G}$ is the classical r matrix. By virtue of the isomorphism $\mathfrak{G}^* \simeq \mathfrak{G}$, we can identify r with a skew-symmetric operator in $\mathfrak{G}$. We require that the operator r satisfy a modified classical Yang—Baxter identity

$$[rX, rY] = r([rX, Y] + [X, rY]) - [X, Y]. \tag{2.8}$$

The bracket on $\mathfrak{G}^* \simeq \mathfrak{G}$, dual to (2.7) is given by

$$[X, Y]_* = \frac{1}{2}([rX, Y] + [X, rY]) \tag{2.9}$$

and by virtue of (2.8) satisfies the Jacobi identity. Let $r_\pm = ½(r\pm\mathrm{id})$. It is readily seen that by virtue of (2.8) $r_\pm: \mathfrak{G}^* \to \mathfrak{G}$ are homomorphisms of Lie algebras. Let $\mathfrak{d} = \mathfrak{G}\oplus\mathfrak{G}$ (direct sum of two copies of $\mathfrak{G}$). Let $\mathfrak{G}^\delta \subset \mathfrak{d}$ be a diagonal subalgebra. The mapping $\mathfrak{G}^* \to \mathfrak{d}$: $X \mapsto (X_+, X_-), X_\pm = r_\pm X$, is an embedding of Lie algebras; thus, we can identify $\mathfrak{G}^*$ with a subalgebra in $\mathfrak{d}$. We equip the algebra with a scalar product, setting

$$\langle\langle (X, X'), (Y, Y')\rangle\rangle = \langle X, Y\rangle - \langle X', Y'\rangle. \tag{2.10}$$

Proposition 2.3. 1. *There is the following decomposition into a direct sum of linear spaces:*

$$\mathfrak{d} = \mathfrak{G}^\delta + \mathfrak{G}^*;$$

2. *Let $P_{\mathfrak{G}^\delta}, P_{\mathfrak{G}^*} \in \mathrm{End}\,\mathfrak{d}$ be projectors onto $\mathfrak{G}^\delta, \mathfrak{G}^* \subset \mathfrak{d}$ parallel to the complementary subalgebra, $r_\mathfrak{d} = P_{\mathfrak{G}^\delta} - P_{\mathfrak{G}^*}$. The operator $r_\mathfrak{d}$ is skew symmetric with respect to the scalar product* (2.10) *and satisfies the identity* (2.8). *Thus, $r_\mathfrak{d}$ defines on $\mathfrak{d}$ the structure of a Lie bialgebra;*

3. *The bialgebra* $(\mathfrak{d}, \mathfrak{d}^*)$ *is canonically isomorphic to the double* $(\mathfrak{G}, \mathfrak{G}^*)$.

An explicit expression for the r matrix $r_\mathfrak{d}$ in terms of the original matrix has the form

$$r_\mathfrak{d} = \begin{pmatrix} r & -2r_+ \\ 2r_- & -r \end{pmatrix} \in \operatorname{End}(\mathfrak{G} \oplus \mathfrak{G}).$$

The bialgebra $(\mathfrak{G}, \mathfrak{G}^*)$ defined by means of an operator $r \in \operatorname{End}\mathfrak{G}$, satisfying the identity (2.8) is called a *factorizable bialgebra*. Thus, the double of a factorizable bialgebra is isomorphic (as a Lie algebra) to the square of $\mathfrak{G}$.

Now let G be a matrix algebraic group with Lie algebra $\mathfrak{G}$, $D=G\times G$. The embedding $\mathfrak{G}^* \subset \mathfrak{d}$ can be extended to a homomorphism $G^* \subset D$; we identify the group G^* with the corresponding subgroup in D. Almost all the elements $(x, y) \in D$ (except for elements of a subset of positive codimension) can be represented in the form

$$(x, y) = (L^+, L^-)(T, T)^{-1}, \tag{2.11}$$

where $(L^+, L^-) \in G^*$, $(T, T) \in G^\delta \subset D$. Let ρ, V be a faithful matrix representation of the group G. The algebra of functions on D is generated by the matrix elements $\rho(x)_{ij}$, $\rho(y)_{ij}$. The matrices $L^\pm$ and T can be regarded as (almost everywhere regular) functions of x and y. Therefore, the matrix elements $\rho(L^\pm)_{ij}$, $\rho(T)_{ij}$ specify a different system of generators of the algebra $\operatorname{Fun}(D)$. The functions $\rho(L^\pm)_{ij}$, $\rho(T)_{ij}$ are rational functions on D whose singularities coincide with the set of elements $(x, y) \in D$ for which the factorization (2.11) does not exist.

It is convenient to specify the Poisson bracket on D on the generators of the ring $\operatorname{Fun} D$. As usual, we use tensor notation, setting $T_1 = T\otimes I$, $T_2 = I\otimes T$, etc.; by definition, the Poisson bracket $\{T_1^V, T_2^V\}$ is the matrix in $\operatorname{End} V \otimes \operatorname{End} V$ whose matrix elements are the Poisson brackets of the functions $\{\rho(T)_{ij}, \rho(T)_{kl}\}$. If no confusion is possible, we shall omit the index V.

Formulas for the Poisson brackets of two systems of generators — (x, y) and $(L^\pm, T)$ — will be helpful. We have

$$\{x_1^V, x_2^V\} = \tfrac{1}{2}(r^V x_1^V x_2^V \pm x_1^V x_2^V r^V), \quad r^V = (\rho\otimes\rho)r, \quad r \in \mathfrak{G}\wedge\mathfrak{G},$$

$$\{y_1^V, y_2^V\} = \tfrac{1}{2}(r^V y_1^V y_2^V \pm y_1^V y_2^V r), \tag{2.12}$$

$$\{y_1^V, x_2^V\} = r_+^V y_1^V x_2^V \pm y_1 x_2 r_+^V, \quad r_+^V = (\rho\otimes\rho)r_+, \quad r_+ \in \mathfrak{G}\otimes\mathfrak{G}.$$

The brackets of the generators $L^\pm$ and T are given by

$$\{T_1^V, T_2^V\}_\pm = \tfrac{1}{2}[r^V, T_1^V T_2^V], \quad \{L_1^\varepsilon, L_2^\varepsilon\}_\pm = \tfrac{1}{2}[r^V, L_1^\varepsilon L_2^\varepsilon], \quad \varepsilon = \pm, \quad \{L_1^+, L_2^-\}_\pm = \tfrac{1}{2}[r^V, L_1^+ L_2^-], \tag{2.13}$$

$$\{L_1^\pm, T_2\}_+ = L_1^\pm T_2 r_\pm, \quad \{L_1^\pm, T_2\}_- = 0. \tag{2.14}$$

Thus, the Poisson structures on $D_\pm$ differ only by the brackets between $L^\pm$ and T. We recall that D_- is a Poisson—Lie group; formulas (2.13) and (2.14) show how the Poisson manifold (but, of course, not as a group) D_- is the direct product of its subgroups G and G^*. The bracket (2.13)—(2.14) is nondegenerate.

The group D_- with bracket (2.12) or (2.13)—(2.14) is a Poisson—Lie group. Therefore, the diagonal mapping Δ: Fun $D_- \to \operatorname{Fun}(D_-\times D_-)$ is a homomorphism of Poisson algebras. This mapping is readily described for the generators x, y:

$$\Delta x = x\dot{\otimes}x, \quad \Delta y = y\dot{\otimes}y \tag{2.15}$$

or, in more detail,

$$\Delta\rho(x)_{ij} = \sum_k \rho(x)_{ik}\otimes\rho(x)_{kj}$$

and, similarly, for $\rho(y)_{ij}$.

It is readily verified that the multiplication $D_-\times D_+ \to D_+$ is a Poisson mapping. Therefore, formula (2.15) is readily transformed into the morphism

$$\operatorname{Fun} D_+ \to \operatorname{Fun} D_- \otimes \operatorname{Fun} D_+,$$

which defines on $\operatorname{Fun}(D_+)$ the structure of a left comodule over $\operatorname{Fun}(D_-)$. We set

$$\Delta x_{(+)} = x_{(-)}\dot{\otimes}x_{(+)}, \quad \Delta y_{(+)} = y_{(-)}\dot{\otimes}y_{(+)} \tag{2.16}$$

(the notation is obvious!).

There is one further group associated with duality theory — the *dual double* D^*. By definition, D^* is the Poisson group corresponding to the Lie bialgebra $(\mathfrak{d}^*, \mathfrak{d})$. Since $\mathfrak{d}^* \simeq \mathfrak{G}\oplus\mathfrak{G}^*$ is a direct sum of Lie algebras, $D^* = G\times G^*$ as a Lie group.

However, the Poisson structure on D^* does not decompose into a product. The ring of functions on D^* is generated by the elements of the matrices $\rho(L^{\pm})$ and $\rho(T)$. We denote these generators by $^*L^{\pm}$ and $^*T$ to distinguish them from the generators of the algebras (2.13)—(2.14). We have

$$\{{}^*T_1, {}^*T_2\} = \tfrac{1}{2}[r, {}^*T_1{}^*T_2], \qquad \{{}^*L_1^{\pm}, {}^*L_2^{\pm}\} = \tfrac{1}{2}[r, {}^*L_1^{\pm}{}^*L_2^{\pm}],$$

$$\{{}^*L_1^{+}, {}^*L_2^{-}\} = [r^{+}, {}^*L_1^{+}{}^*L_2^{-}], \qquad \{{}^*L_1^{\pm}, {}^*T_2\} = [r^{\pm}, {}^*L_1^{\pm}{}^*T_2]. \tag{2.17}$$

The coproduct in the algebra Fun(D^*) is given by

$$\Delta{}^*T = {}^*T\dot{\otimes}{}^*T, \quad \Delta{}^*L^{\pm} = {}^*L^{\pm}\dot{\otimes}{}^*L^{\pm}. \tag{2.18}$$

We turn to the study of what is for us the main bracket $\{,\}_+$ on D and consider in more detail the connection between the different systems of generators of the ring Fun(D_+). We note first of all that the decomposition (2.11) enables us to regard $L^{\pm}$ as the generators of the ring Fun(D/G^{δ}) and T as generators of the ring Fun($G^*\backslash D$). We can consider the decomposition

$$(x, y) = ('T, 'T)('L^{+}, 'L^{-})^{-1}, \tag{2.19}$$

in which the elements G^{δ}, G are multiplied in the reverse order. It is obvious that the matrix elements of $'T$ generate the ring Fun(D/G^*), and the matrix elements of $'L^{\pm}$ the ring Fun($G\backslash D$). As we know, the canonical projections

$$\begin{array}{ccc} & D & \\ {\scriptstyle\pi}\swarrow & & \searrow{\scriptstyle\pi'} \\ D/G & & G\backslash D \end{array}, \qquad \begin{array}{ccc} & D & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle p'} \\ D/G^* & & G^*\backslash D \end{array} \tag{2.20}$$

form dual pairs. Thus, $'L^{\pm}$ and $'T$ satisfy the relations (2.13) and (2.14), and also

$$\{L_1^{\pm}, 'L_2^{\pm}\} = \{T_1, 'T_2\} = 0. \tag{2.21}$$

The decompositions (2.11) and (2.19) are the analog of the definition of the right and left moments in solid-state theory or of the chiral decomposition in field theory (with regard to this last analogy, see [16]).

The factor spaces D/G, $G\backslash D$ can be canonically identified with the group G itself. The projections π, π' are given by the formulas $\pi(x, y) = xy^{-1}$, $\pi'(x, y) = y^{-1}x$. Accordingly, in the rings of functions Fun(D/G), Fun($G\backslash D$) other generators can be chosen. We set $L = \rho(xy^{-1})$, $'L = \rho(y^{-1}x)$ (where, as above, ρ is a faithful matrix representation of the group G). Obviously,

$$L = L^{+}(L^{-})^{-1}, \quad 'L = ('L^{+})^{-1}{}'L^{-}.$$

The Poisson brackets of these generators are given by

$$\{L_1, L_2\} = L_1 r_+ L_2 + L_2 r_- L_1 - \tfrac{1}{2} r L_1 L_2 - \tfrac{1}{2} L_1 L_2 r, \tag{2.22}$$

$$\{L_1, T_2\} = L_1 T_2 r_- - T_2 r_+ L_1. \tag{2.23}$$

In accordance with the general theory, the symplectic sheets in $D/G \simeq G$ are connected components of the sets $\pi(\pi'^{-1}(x))$, $x \in G\backslash D$. In the given case, $\pi(x, y) = xy^{-1}$, $\pi'(x, y) = y^{-1}x$. Thus, the symplectic sheets in $G \simeq D/G$ are simply the classes of conjugate elements, and the action of the group G on itself by conjugations gives the Poisson mapping

$$G \times G \to G : (x, L) \mapsto xLx^{-1}. \tag{2.24}$$

Here, the Poisson bracket $\{L_1, L_2\}$ is given by (2.22), and the bracket of the matrix elements of x by the formula $\{x_1, x_2\} = ½[r, x_1x_2]$. By duality, we obtain the morphism of Poisson algebras

$$\mathrm{Fun}(G) \to \mathrm{Fun}(G) \otimes \mathrm{Fun}(G) : \quad L \mapsto x_1 L_2 x_1^{-1}. \tag{2.25}$$

The action (2.22) and its dual coaction (2.25) provide a special case of so-called *dressing transformations* [12].

Note in conclusion that the Casimir functions of the algebra (2.22), i.e., the center of the Poisson algebra Fun(D/G), are simply the central functions on the group G. The generators of the ring of Casimir functions have the form

$$c_k = \mathrm{tr}\, L^k = \mathrm{tr}\, 'L^k. \tag{2.26}$$

3. QUANTIZATION

Let A be a quasitriangular factorizable Hopf algebra [1,15]. By definition, the condition on A means the following. There is defined an element $R \in A \otimes A$ (*universal R matrix*) that satisfies the relations*

$$\Delta'(x) = R\Delta(x)R^{-1} \quad (\Delta'(x) = P(\Delta(x))) \ , \tag{3.1}$$

$$(\Delta \otimes \mathrm{id})R = R_{13}R_{23}, \quad (\mathrm{id} \otimes \Delta)R = R_{13}R_{12}, \quad (\varepsilon \otimes \mathrm{id})R = (\mathrm{id} \otimes \varepsilon)R = 1. \tag{3.2}$$

Under these assumptions, the element $R \in A \otimes A$ is invertible and $R^{-1} = (S \otimes \mathrm{id})R$, where S is the antipode of A. We set $R_+ = R$, $R_- = P(R^{-1})$. Let A^* be the dual Hopf algebra, and A^0 be the same algebra with opposite comultiplication. We define mappings by the formulas $R^{\pm}$: $A^0 \to A$:

$$R^{\pm} : f \mapsto \langle f \otimes \mathrm{id}, R_{\pm} \rangle. \tag{3.3}$$

It follows from the relations (3.2) that $R^{\pm}$ are homomorphisms of Hopf algebras. We consider the combined mapping

$$A^0 \xrightarrow{(R^+ \otimes R^-)\Delta^0} A \otimes A \xrightarrow{m(\mathrm{id} \otimes S^{-1})} A. \tag{3.4}$$

The quasitriangular algebra A is said to be *factorizable* if the composite mapping is an isomorphism of linear spaces.

Let $\{e_i\}$ be a linear basis in A, and $\{e^i\}$ be the dual basis in A^0. Let

$$T = e^i \otimes e_i \in A^0 \otimes A \tag{3.5}$$

be a canonical element. In accordance with the rules of tensor notation, we set

$$T_1 T_2 = e^i e^j \otimes e_i \otimes e_j \in A^0 \otimes A \otimes A, \quad T_2 T_1 = e^j e^i \otimes e_i \otimes e_j \in A^0 \otimes A \otimes A.$$

From (3.1)

$$T_2 T_1 = R_{\pm} T_1 T_2 R_{\pm}^{-1}. \tag{3.6}$$

Let

$$L^{\pm} = (R^{\pm} \otimes \mathrm{id})T \in A \otimes A. \tag{3.7}$$

We have

$$L_2^{\pm} L_1^{\pm} = R_+ L_1^{\pm} L_2^{\pm} R_+^{-1}, \quad L_2^- L_1^+ = R_+ L_1^+ L_2^- R_+^{-1}. \tag{3.8}$$

We set $L = L^+(L^-)^{-1}$. Then

$$R_+^{-1} L_2 R_+ L_1 = L_1 R_-^{-1} L_2 R_-. \tag{3.9}$$

Now suppose ρ_V and ρ_W are representations of A in the spaces V and W. We set $R^{VW} = (\rho_V \otimes \rho_W)R$. Let

$$T^V = (\mathrm{id} \otimes \rho_V)T \in A^0 \otimes \mathrm{End}\, V, \quad L^{\pm V} = (\mathrm{id} \otimes \rho_V)L^{\pm} \in A \otimes \mathrm{End}\, V.$$

The relations (3.6)—(3.9) become

$$T_2^W T_1^V = R_{\pm}^{VW} T_1^V T_2^W (R_{\pm}^{VW})^{-1}, \quad L_2^{\pm W} L_1^{\pm V} = R_+^{VW} L_1^{\pm V} L_2^{\pm W} (R_+^{VW})^{-1},$$

$$(R_+^{VW})^{-1} L_2^W R_+^{VW} L_1^V = L_1^V (R_-^{VW})^{-1} L_2^W R_-^{VW}.$$

We now assume that the algebra A is semiclassical, i.e., is defined over $\mathbb{C}[[h]]$, where h is the deformation parameter, and A/hA (as Hopf algebra) is identical to the universal covering algebra of some Lie algebra $\mathfrak{G}$, $R_{\pm} = 1 + hr_{\pm} + o(h)$, where $r_{\pm} \in U(\mathfrak{G})^{\otimes 2}$. It is easy to show that in reality $r_{\pm} \in \mathfrak{G} \otimes \mathfrak{G}$, where $\mathfrak{G}$ can be identified with the subalgebra of primitive elements in $U(\mathfrak{G})$, and $r_{\pm}$ satisfy the classical Yang—Baxter equation. In the semiclassical limit, the factorization mapping (3.4) generates an isomorphism $\mathfrak{G}^* \to \mathfrak{G}$, i.e., an invariant scalar product on $\mathfrak{G}$; thus, the algebra $\mathfrak{G}$ inherits the structure of a factorizable Lie bialgebra. Therefore, there are defined the groups G, G^* and the Poisson algebras of functions Fun(G), Fun(G^*). We assume that G, G^* are matrix algebraic groups and that their faithful representation in the space V is consistent with the representation ρ_V of the algebra A.

*As usual, P is the transposition operator, $P(x \otimes y) = y \otimes x$.

Proposition 3.1. *The associative algebras* $\mathrm{Fun}_q(G)$, $\mathrm{Fun}_q(G^*)$ *generated, respectively, by the matrix elements of the matrices* T^V *and* $L^{\pm V}$ *or* L^V *and the relations (3.8) are quantizations of the Poisson algebras* $\mathrm{Fun}(G)$, $\mathrm{Fun}(G^*)$ *with relations* (2.13) *and* (2.22).

The construction of the operators T, $L^{\pm}$ in terms of the canonical element and the universal R matrix give the homomorphisms $A \to \mathrm{Fun}_q(G^*)$, $A^* \to \mathrm{Fun}_q(G)$. In fact, if ρ is a faithful representation, it is easy to show that these mappings are indeed isomorphisms. If $A = U_q(\mathfrak{G})$, then, using the explicit expressions for the universal R matrix [9,10], we can express the generators $L^{\pm}$, T in terms of the Drinfel'd—Jimbo generators (for $\mathfrak{G} = sl(2)$ these formulas are elementary and are given in [3]).

We define the action $A \otimes A^* \to A^*$ by "left differentiations":

$$x \otimes f \mapsto D_x f = \langle x \otimes \mathrm{id}, \Delta f \rangle. \tag{3.10}$$

Obviously,

$$\langle x, f \rangle = \varepsilon(D_x f), \tag{3.11}$$

i.e., the canonical pairing between A and A^* is given by the "value of the derivative at the identity." We consider the algebra of operators in A^* generated by the operators of multiplication by elements $f \in A^*$ and by differential operators D_x, $x \in A$. It is convenient to express the commutation relations in this algebra by considering the operators

$$D_{L^{\pm}} = (D \otimes \mathrm{id}) L^{\pm} \in \mathrm{End}\, A^* \otimes A.$$

Obviously, the mapping D: $A \to \mathrm{End}\, A^*$ is a homomorphism of algebras, and therefore it is sufficient to calculate the commutation relations between $D_L{\pm}$ or D_L and T.

Proposition 3.2.

$$D_{L_1^{\pm}} T_2 = T_2 D_{L_1^{\pm}} R_{\pm}, \quad D_{L_1} T_2 = T_2 R_+ D_{L_1} R_-^{-1}, \quad \langle L^{\pm}, T \rangle = R_{\pm}. \tag{3.12}$$

To interpret these relations, we consider the algebra generated by the generators T_{ij}^V, $L_{ij}^{\pm V}$ (or L_{ij}^V) and the relations (4.8) augmented by the relations

$$L_1^{\pm V} T_2^W = T_2^W L_1^{\pm V} R_{\pm}^{VW}, \quad L_1^V T_2^W = T_2^W R_+^{VW} L_1^V (R_-^{VW})^{-1}. \tag{3.13}$$

Proposition 3.3. **1**. *The algebra* (3.8)—(3.13) *is a quantization of the Poisson algebra* $\mathrm{Fun}(D_+)$. **2**. *The realization of* $L_{\pm}$, T *by operators of differentiation and multiplication in* A^* *gives a representation of this algebra in the space* $\mathrm{End}\, A$.

It is natural to denote the algebra (3.8)—(3.13) by $\mathrm{Fun}_q(D_+)$; we shall call it the *Heisenberg double of* A. Multiplying the matrices of the generators $L^{\pm}$, T, we can arrive at a more symmetric realization of the Heisenberg double.

Proposition 3.4. *Let* $X = L^+ T^{-1}$, $Y = L^- T^{-1}$. *We have the commutation relations*

$$R_+ X_1 X_2 = X_2 X_1 R_-^{-1}, \quad R_+ Y_1 Y_2 = Y_2 Y_1 R_-^{-1}, \quad R_+ X_1 Y_2 = Y_2 X_1 R_+^{-1}. \tag{3.14}$$

The algebra (3.14) *is a quantization of the Poisson algebra* (2.12).

Besides the Heisenberg double, there is also defined the *Drinfel'd double* of the algebra A [1]. We recall that the double of the Lie bialgebra $(\mathfrak{G}, \mathfrak{G}^*)$ is the bialgebra $(\mathfrak{d}, \mathfrak{d}^*)$. Let D, D^* be the corresponding Poisson—Lie groups. The Drinfel'd double is the canonically defined quasitriangular factorizable Hopf algebra $\mathcal{D}$, which is a quantization of the bialgebra $(\mathfrak{d}, \mathfrak{d}^*)$. In accordance with the quantum duality principle, $\mathcal{D} = \mathrm{Fun}_q(D^*)$, and the dual Hopf algebra $\mathcal{D}^* = \mathrm{Fun}_q(D_-)$.

We first describe the last algebra in terms of generators and relations. Comparison of the expressions for the Poisson brackets $\{,\}_{\pm}$ on the group $\mathcal{D}$ shows that the commutation relations between the elements of $L^{\pm}$ and between the elements of T are the same as in the algebra $\mathrm{Fun}_q(D_+)$; the crossed commutators are zero.

Proposition 3.5. *The associative algebra* $\mathrm{Fun}_q(D_-)$ *generated by the matrix elements of the matrices* $\mathcal{L}^{\pm}$, $\mathcal{T}$ *with the relations*

$$RT_1T_2 = T_2T_1R, \quad R_+ \mathcal{L}_1^{\pm} \mathcal{L}_2^{\pm} = \mathcal{L}_2^{\pm} \mathcal{L}_1^{\pm} R_+, \quad R_+ \mathcal{L}_1^+ \mathcal{L}_2^- = \mathcal{L}_2^- \mathcal{L}_1^+ R_+, \quad T_1 \mathcal{L}_2^{\pm} = \mathcal{L}_2^{\pm} T_1, \tag{3.15}$$

is isomorphic (as an algebra) to the dual Drinfel'd double $\mathcal{D}^*$.

To describe the coproduct in the algebra $\mathrm{Fun}_q(D_-)$, it is convenient to go over to a different system of generators [this transition is suggested by formulas (2.12)]. We set

$${}^{(-)}X = \mathcal{L}^+ \mathcal{T}, \quad {}^{(-)}Y = \mathcal{L}^- \mathcal{T}. \tag{3.16}$$

Proposition 3.6. **1**. *The matrices* ${}^{(-)}X$, ${}^{(-)}Y$ *satisfy the commutation relations*

$$ {}^{(-)}X_2{}^{(-)}X_1 = R_+{}^{(-)}X_1{}^{(-)}X_2R_+^{-1}, \quad {}^{(-)}Y_2{}^{(-)}Y_1 = R_+{}^{(-)}Y_1{}^{(-)}Y_2R_+^{-1}, \quad {}^{(-)}Y_2{}^{(-)}X_1 = R_+{}^{(-)}X_1{}^{(-)}Y_2R_+^{-1}. \quad (3.17) $$

2. *The formulas*

$$ \Delta X = {}^{(-)}X\dot{\otimes}{}^{(-)}X, \ \Delta Y = {}^{(-)}Y\dot{\otimes}{}^{(-)}Y $$

define on $\mathrm{Fun}_q(D_-)$ *the structure of a Hopf algebra; this algebra is isomorphic to the dual Drinfel'd double* $\mathcal{D}^*$.

3. *The formulas*

$$ \Delta_+X = {}^{(-)}X\dot{\otimes}X, \qquad \Delta_+Y = {}^{(-)}Y\dot{\otimes}Y $$

define on the algebra $\mathrm{Fun}_q(D_+)$ *the structure of a left* $\mathcal{D}^*$ *comodule.*

The algebra $\mathrm{Fun}_q(D^*)$ admits a similar description.

Proposition 3.7. *The associative algebra* $\mathrm{Fun}_q(D^*)$ *generated by the matrix elements of the matrices* ${}^*\mathcal{L}^\pm$, ${}^*T$ *with relations*

$$ {}^*T_2{}^*T_1 = R^*T_1{}^*T_2R^{-1}, \qquad {}^*\mathcal{L}_2^{\pm *}\mathcal{L}_1^\pm = R_+{}^*\mathcal{L}_1^{\pm *}\mathcal{L}_2^\pm R_+^{-1}, $$

$$ {}^*\mathcal{L}_2^{- *}\mathcal{L}_1^+ = R_+{}^*\mathcal{L}_1^{+ *}\mathcal{L}_2^- R_+^{-1}, \qquad {}^*\mathcal{L}_2^{\pm *}T_1 = R_\pm{}^*T_1{}^*\mathcal{L}_2^\pm R_\pm^{-1} $$

and with coproduct

$$ \Delta^*T = {}^*T\dot{\otimes}{}^*T, \qquad \Delta^*\mathcal{L}^\pm = {}^*\mathcal{L}^\pm\dot{\otimes}{}^*\mathcal{L}^\pm, $$

is isomorphic to the Drinfel'd double $\mathcal{D}$ *as a Hopf algebra.*

The algebra $\mathrm{Fun}_q(D^*)$ also admits a natural "operator" realization. We first define the (right) adjoint action of the Hopf algebra A on itself:

$$ \mathrm{Ad} : A \otimes A \to A : x \otimes y \to Sy_i^{(1)}xy_i^{(2)}, \tag{3.18} $$

where $\Delta y = y_i^{(1)} \otimes y_i^{(2)}$ is the coproduct in A.

The coadjoint action Ad^*: $A^* \otimes A \to A$ is defined by

$$ \langle \mathrm{Ad}^*(f \otimes x), y\rangle = \langle f, \mathrm{Ad}(x \otimes y)\rangle. \tag{3.19} $$

We have

$$ \mathrm{Ad}^*(f \otimes x) = \langle Sf_j^{(1)}f_j^{(2)}, x\rangle f_j^{(2)}, \tag{3.20} $$

where $\Delta^{(2)}f = f_j^{(1)} \otimes f_j^{(2)} \otimes f_j^{(3)}$.

We consider the algebra of operators in $A^* = \mathrm{Fun}_q(G)$ generated by the operators of multiplication $m(f)$, $f \in A^*$, and by the operators $\mathrm{Ad}^*\, x$, $x \in A = \mathrm{Fun}_q(G^*)$. Let

$$ {}^*\mathcal{L}^{\pm V} = (\mathrm{Ad}^* \otimes \mathrm{id}_v)L^{\pm V}, \ T^V = (m \otimes \mathrm{id})T^V, $$

where $L^{\pm V}$, T^V have the same meaning as in (3.6) and (3.5).

Proposition 3.8. *The relations between the operator matrices* ${}^*\mathcal{L}^\pm$, ${}^*T$ *have the form*

$$ {}^*\mathcal{L}_2^{\pm *}T_1 = R_\pm{}^*T_1{}^*\mathcal{L}_2^\pm R_\pm^{-1}. \tag{3.21} $$

Thus, the algebra generated by the matrix elements of the operator matrices ${}^*\mathcal{L}^{\pm V}$, ${}^*T^V$ *is a representation of the algebra* $\mathrm{Fun}_q(D_+)$.

We now describe the quantum analog of the Poisson reduction and of the chiral decomposition for the algebra $\mathrm{Fun}_q(D_+)$ [cf. (2.20)—(2.22)]. We have seen above that the algebra $\mathrm{Fun}_q(D_+)$ has the natural structure of the left $\mathrm{Fun}_q(D_-)$ comodule. Similarly, there is also defined the structure of the right $\mathrm{Fun}_q(D_-)$ comodule. We can specialize these formulas and obtain on $\mathrm{Fun}_q(D_+)$ the structure of the left and right $\mathrm{Fun}_q(G)$ comodule. Namely, we set

$$ \Delta_L X = T\dot{\otimes}X, \quad \Delta_L Y = T\dot{\otimes}Y, \quad \Delta_R X = X\dot{\otimes}T^{-1}, \quad \Delta_R Y = Y\dot{\otimes}T^{-1}, \tag{3.22} $$

where T is the matrix of generators of the algebra $\mathrm{Fun}_q(G)$, $T^{-1} = (S \otimes \mathrm{id})T$, and S is the antipode of the algebra $\mathrm{Fun}_q(G)$. The necessary formal properties of the homomorphisms Δ_L, Δ_R are directly verified. Now let $\mathrm{Fun}_q(G\backslash D)$, $\mathrm{Fun}_q(D/G) \subset \mathrm{Fun}_q(D_+)$ be subalgebras of left- (right-) invariant functions. By definition,

$$ \varphi \in \mathrm{Fun}_q(G\backslash D) \Longleftrightarrow \Delta_L\varphi \in 1 \otimes \mathrm{Fun}_q(D_+), \qquad \psi \in \mathrm{Fun}_q(D/G) \Longleftrightarrow \Delta_R\psi \in \mathrm{Fun}_q(D_+) \otimes 1. $$

Proposition 3.9. *The algebra* $\mathrm{Fun}_q(G\backslash D)$ *is generated by the matrix elements of the matrix* $'L = X^{-1}Y$; *similarly, the* *algebra* $\mathrm{Fun}_q(D/G)$ *is generated by the matrix elements of the matrix* $L = XY^{-1}$. *The commutation relations of the matrices* L, $'L$ *are given by* (3.9); *in addition,* $L_1' L_2 = {}'L_2 L_1$, *i.e., the subalgebras* $\mathrm{Fun}_q(D/G)$, $\mathrm{Fun}_q(G\backslash D)$ *centralize each other. Thus,* $\mathrm{Fun}_q(D/G) \simeq \mathrm{Fun}_q(G^*)$.

Obviously, the algebra $\mathrm{Fun}_q(D/G)$ preserves the structure of the left $\mathrm{Fun}_q(G)$ comodule defined by

$$\Delta L = T_1 L_2 T_1^{-1} \tag{3.23}$$

(or, in detail,

$$\Delta L_{ij} = T_{ik} T_{lj}^{-1} \otimes L_{kl}).$$

Formula (3.23) defines in the given situation the quantum analog of "dressing transformations." The following simple proposition is the analog of the description of the center of the Poisson algebra $\mathrm{Fun}_q(G)$ at the end of Sec. 2 [see (2.24)—(2.26)] and simultaneously Gel'fand's well-known theorem on the center of the universal covering algebra.

Proposition 3.10. *The center of the algebra* $\mathrm{Fun}_q(D/G) \simeq \mathrm{Fun}_q(G^*)$ *is identical to the subalgebra of invariants of the coaction* (3.23), *i.e., functions* $\varphi \in \mathrm{Fun}_q(D/G)$ *for which* $\Delta\varphi \in 1 \otimes \mathrm{Fun}_q(D/G)$.

We shall consider in more detail the quantum dressing transformations in a separate paper.

4. HEISENBERG DOUBLE AND QUANTUM FOURIER TRANSFORMATION

In studying the commutation relations for the operators $L^{\pm}$, T we have hitherto fixed the representation (ρ, V). If we consider arbitrary irreducible representations and their direct sums, then the quantum duality principle can be related to the intuitively attractive *Fourier transformation* [see formula (1.1)].

Let A be a Hopf algebra, and A^* be the dual Hopf algebra. Let $\hat{A} = \mathrm{Spec}\, A$ be the set of irreducible representations of A. We do not introduce on A any star structure, i.e., we do not assume that the representations $\lambda \in \hat{A}$ are unitary. We require $\hat{A}$ to be closed with respect to the tensor product (this condition is formulated more precisely below). For $\lambda \in \hat{A}$, let V_λ be the corresponding irreducible A module.

Let Fun A be the space of functions on the spectrum such that $\varphi_\lambda \in \mathrm{End}\, V_\lambda$. We shall represent the dual space as a space of "matrix-valued measures" on $\hat{A}$. For $\varphi \in \mathrm{Fun}\, \hat{A}$, $\sigma = d\sigma(\lambda) \in \mathrm{Mes}\, \hat{A}$, we set

$$\langle \sigma, \varphi \rangle = \int_{\hat{A}} \mathrm{tr}_{V_\lambda} (\varphi(\lambda)\, d\sigma(\lambda)). \tag{4.1}$$

The condition that $\hat{A}$ be closed with respect to the tensor product takes the form that for all π_1, $\pi_2 \in \hat{A}$, $x \in A$

$$\pi_1 \otimes \pi_2(x) = \int_{\hat{A}} \otimes\, \mathrm{tr}_{V_\lambda} (\lambda(x)\, dC(\lambda; \pi_1, \pi_2)), \tag{4.2}$$

where $dC(\lambda;\ \pi_1,\ \pi_2) \in \mathrm{Mes}\, \hat{A} \otimes \mathrm{Fun}\, \hat{A} \otimes \mathrm{Fun}\, \hat{A}$ are Clebsch—Gordan coefficients.

Similarly, we define $\hat{A}^* = \mathrm{Spec}\, A^*$ and the spaces Fun $\hat{A}^*$, Mes $\hat{A}^*$.

We denote by $\mathcal{R}$ the canonical element in $A^* \otimes A$ (we change somewhat the notation introduced in Sec. 3) and set

$$T_\rho = (\mathrm{id} \otimes \rho)\mathcal{R}, \quad \rho \in \hat{A}, \qquad L_\lambda = (\lambda \otimes \mathrm{id})\mathcal{R}, \quad \lambda \in \hat{A}^*. \tag{4.3}$$

We define the spectral representation of T: Mes $\hat{A} \to A^*$ by

$$T(\varphi) = \int_{\hat{A}} \mathrm{tr}_{V_\rho} (T_\rho\, d\varphi(\rho)). \tag{4.4}$$

Similarly, the spectral representation of L: Mes $\hat{A}^* \to A$ is defined by

$$L(\psi) = \int_{\hat{A}^*} \mathrm{tr}_{W_\lambda} (L_\lambda\, d\psi(\lambda)). \tag{4.5}$$

We define convolutions of measures:

$$* : \mathrm{Mes}\, \hat{A} \otimes \mathrm{Mes}\, \hat{A} \to \mathrm{Mes}\, \hat{A}, \qquad * : \mathrm{Mes}\, \hat{A}^* \otimes \mathrm{Mes}\, \hat{A}^* \to \mathrm{Mes}\, \hat{A}^*,$$

requiring that

$$T(\varphi_1 * \varphi_2) = T(\varphi_1) T(\varphi_2), \qquad L(\psi_1 * \psi_2) = L(\psi_1) L(\psi_2). \tag{4.6}$$

Proposition 4.1. **1.** *The convolution of measures on $\hat{A}$ is defined by*

$$\varphi_1 * \varphi_2(\rho) \int_{\widehat{A}\times\widehat{A}} \mathrm{tr}_{V_{\rho_1}\otimes V_{\rho_2}} \big(C(\rho;\rho_1,\rho_2)\, d\varphi_1(\rho_1)\, d\varphi_2(\rho_2)\big),$$

where $C(\rho;\ \rho_1,\ \rho_2)$ are the Clebsch—Gordan coefficients of the algebra A.

2. *Similarly, the convolution of measures on $\hat{A}^*$ is defined by*

$$\psi_1 * \psi_2(\lambda) = \int_{\widehat{A}^*\times\widehat{A}^*} \mathrm{tr}_{W_{\lambda_1}\otimes W_{\lambda_2}} \big(C^*(\lambda;\lambda_1,\lambda_2)\, d\psi_1(\lambda_1)\, d\psi_2(\lambda_2)\big),$$

where $C^*(\lambda;\ \lambda_1,\ \lambda_2)$ are the Clebsch—Gordan coefficients of the algebra A^*.

In terms of spectral representations, the pairing $A^*\otimes A\rightarrow\mathbf{C}$ takes the form

$$\langle\varphi \mid \psi\rangle = \int_{\widehat{A}\times\widehat{A}^*} \mathrm{tr}_{V_\rho\otimes W_\lambda} \big(R_{\lambda\rho}\, d\varphi(\rho)\, d\psi(\lambda)\big), \tag{4.7}$$

where

$$R_{\lambda\rho} = (\lambda\otimes\rho)\mathcal{R}. \tag{4.8}$$

We define the Fourier transformation

$$\Phi : \mathrm{Mes}\,\widehat{A}^* \longrightarrow \mathrm{Fun}\,\widehat{A},$$

setting

$$(\Phi\psi)(\rho) = \int_{\widehat{A}^*} \mathrm{tr}_{W_\lambda}\big(R_{\lambda\rho}\, d\psi(\lambda)\big). \tag{4.9}$$

Proposition 4.2. $\Phi(\psi_1 * \psi_2) = \Phi\psi_1\Phi\psi_2$.

Similarly, we define the conjugate transformation

$$\Phi^\wedge : \mathrm{Mes}\,\widehat{A} \longrightarrow \mathrm{Fun}\,\widehat{A}^*.$$

Example. Let $A=S(V)$ be a symmetric algebra of the space V. Then $A^*=S(V^*)$. We choose a basis $\{e_i\}$ in V, and let $\{e^i\}$ be the dual basis in V^*. Then the canonical element in $A^*\otimes A$ is given by

$$\mathcal{R} = \exp(e^i\otimes e_i).$$

The irreducible representations of A and A^* are one dimensional, and we can identify Spec A with the space V^*; for the dual, Spec $A^*=V$. Thus, the convolutions (4.6) are transformed into ordinary convolutions of measures in a linear space, and the Fourier transformation (4.9) into an ordinary Fourier—Laplace transformation.

We define the action of Mes $\hat{A}^*$ on Mes $\hat{A}$ by

$$\psi\otimes\varphi \longrightarrow \Phi\psi\cdot\varphi.$$

Proposition 4.3. *The algebra of operators on* Mes $\hat{A}$ *generated by the operators of convolution with measures* $\varphi\in$ Mes $\hat{A}$ *and operators of multiplication by* $\Phi\psi$, $\psi\in$ Mes $\hat{A}^*$, *is a representation of the algebra* $\mathrm{Fun}_q(D_+)$.

If A is a quasitriangular Hopf algebra, it is easy to define a variant of the theory of Fourier transformation with kernels $R_{\pm}^{\lambda\mu}$, λ, $\mu\in$ Spec A. Because of the lack of space, we shall not dwell on this.

5. TWISTED DOUBLE. DEFORMATIONS OF CLASSICAL AND QUANTUM ALGEBRAS

We turn to the consideration of Poisson algebras of functions and consider their deformations. We first consider a simple example — deformation of the Lie—Poisson bracket on the dual space to the Lie algebra. Let $\mathfrak{G}$ be a Lie algebra such that $H^2(\mathfrak{G})\neq 0$. We choose a nontrivial 2-cocycle $\omega\in C^2(\mathfrak{G})$, and let $\widehat{\mathfrak{G}}_\omega = \mathfrak{G}\dotplus\mathbb{R}$ be the associated central extension of $\mathfrak{G}$. The dual space $\widehat{\mathfrak{G}}^*_\omega$ can be identified with $\mathfrak{G}^*\oplus\mathbb{R}$. The variable $c\in\mathbb{R}$ lies in the center of the Poisson algebra $\mathrm{Fun}(\widehat{\mathfrak{G}}^*_\omega)$. Having fixed c, we obtain a one-parameter family of Poisson algebras $\mathrm{Fun}(\mathfrak{G}^*)_c$, which can be regarded as deformations of the Poisson algebra $\mathrm{Fun}(\mathfrak{G}^*)$. Obviously, the "universal deformation" of the algebra $\mathrm{Fun}(\mathfrak{G}^*)$ is parametrized by the elements of the group $H^2(\mathfrak{G})$.

For Poisson groups, one can pose the similar problem of the deformations of the Poisson algebras of functions. We first consider how in this situation the deformations associated with the central extensions of groups are described [17].

Suppose, as in Sec. 2, $(\mathfrak{G}, \mathfrak{G}^*)$ is a factorizable Lie bialgebra, $\partial \in \mathrm{Der}(\mathfrak{G}, \mathfrak{G}^*)$. By definition, ∂ is a differentiation of the algebra $\mathfrak{G}$ that is skew symmetric with respect to the scalar product on $\mathfrak{G}$ and commutes with an operator $r \in \mathrm{End}\,\mathfrak{G}$. The formula

$$\omega(X,Y) = (X, \partial Y) \tag{5.1}$$

defines a 2-cocycle on $\mathfrak{G}$; thus, we obtain a homomorphism $\mathrm{Der}(\mathfrak{G}, \mathfrak{G}^*) \to C^2(\mathfrak{G})$. In the typical situation, $\mathfrak{G} = L\mathfrak{A}$ is a loop algebra with standard "trigonometric" r matrix, and $\partial = \partial_x$ is the derivative with respect to the loop parameter. In this case, the group of classes $[\mathrm{Der}(\mathfrak{G}, \mathfrak{G}^*)]$ (i.e., the factor group of all differentiations with respect to the module of the internal ones) is isomorphic to $H^2(L\mathfrak{A})$. For simple $\mathfrak{A}$, the group $H^2(L\mathfrak{A}) = \mathbb{R}$ is generated by the cocycle (5.1).

Let $\widehat{\mathfrak{G}} = \mathfrak{G} + \mathbb{R}c$ be the central extension of the algebra $\mathfrak{G}$ generated by the cocycle (5.1). It can be shown that $\widehat{\mathfrak{G}}$ has the canonical structure of a Lie bialgebra, and the dual algebra $\widehat{\mathfrak{G}}^*$ is the semidirect product

$$\widehat{\mathfrak{G}}^* = \mathfrak{G}^* + \mathbb{R}\partial.$$

More precisely, we define in $\widehat{\mathfrak{G}}^*$ the commutator by

$$[f + \alpha\partial, g + \beta\partial] = [f,g]_* + \alpha\partial r(g) - \beta\partial r(f). \tag{5.2}$$

Proposition 5.1. *The pair $(\widehat{\mathfrak{G}}, \widehat{\mathfrak{G}}^*)$ is a Lie bialgebra.*

It is easy to describe the double of the bialgebra $(\widehat{\mathfrak{G}}, \widehat{\mathfrak{G}}^*)$. We do this under the simplifying assumption

$$\partial - \partial \circ r^2 = 0 \tag{5.3}$$

(this condition is satisfied for standard trigonometric r matrices on loop algebras). Let

$$\mathfrak{d} = \mathfrak{G} \oplus \mathfrak{G}, \quad \widehat{\mathfrak{d}} = \mathfrak{d} + \mathbb{R}c + \mathbb{R}\partial.$$

We extend the differentiation ∂ to $\mathfrak{G} \oplus \mathfrak{G}$ by

$$\partial\begin{pmatrix} X \\ X' \end{pmatrix} = \begin{pmatrix} \partial X \\ -\partial X' \end{pmatrix},$$

and we define the cocycle $\omega_{\mathfrak{d}}$, which defines the "central" component of the Lie bracket in $\widehat{\mathfrak{d}}$, by

$$\omega_{\mathfrak{d}}\left(\begin{pmatrix} X \\ X' \end{pmatrix}, \begin{pmatrix} Y \\ Y' \end{pmatrix}\right) = \frac{1}{2}\left\langle\left\langle\begin{pmatrix} X \\ X' \end{pmatrix}, \begin{pmatrix} \partial & 0 \\ 0 & -\partial \end{pmatrix}\right\rangle\right\rangle. \tag{5.4}$$

Proposition 5.2. *Suppose the condition* (5.3) *holds. Then the algebra $\widehat{\mathfrak{d}}$ is isomorphic to the double $(\widehat{\mathfrak{G}}, \widehat{\mathfrak{G}}^*)$; the embedding $\widehat{\mathfrak{G}}, \widehat{\mathfrak{G}}^* \hookrightarrow \mathfrak{d}$ is given by*

$$(X, \alpha) \mapsto (X, X, \alpha c, 0), \quad (f, \beta) \mapsto (f_+, f_-, 0, \beta\partial).$$

(By virtue of the condition (5.3), the cocycle (5.4) vanishes on restriction to $\widehat{\mathfrak{G}}^* \subset \mathfrak{d}$.)

Let C be the group of automorphisms of $(\mathfrak{G}, \mathfrak{G}^*)$ generated by the differentiation ∂. The Lie group corresponding to the algebra $\widehat{\mathfrak{G}}^*$ is the semidirect product $G^* \ltimes C$. In accordance with the quantum duality principle, the quantum covering algebra $U_q(\widehat{\mathfrak{G}})$ can be identified with $\mathrm{Fun}_q(\hat{G}^*)$. As a first step to the study of the quantum algebra, we consider the Poisson algebra of functions on $\hat{G}^*$.

Proposition 5.3. *Let c be an affine coordinate on the group $C \simeq \mathbb{R}^x$. The function c lies in the center of the Poisson algebra* $\mathrm{Fun}(\hat{G}^*)$.

The variable c has the meaning of a central charge. Proposition 5.3 means that the bracket on $\hat{G}^*$ depends on $c \in \mathbb{R}^x$ as on a parameter, i.e., we obtain a one-parameter family of Poisson brackets $\{,\}_c$ on the group G^*. After quantization, it is associated with a one-parameter family of algebras

$$U_q(\mathfrak{G})_c = U_q(\widehat{\mathfrak{G}})/(c = \mathrm{const}), \tag{5.5}$$

the factor group of the algebra $U_q(\widehat{\mathfrak{G}})$, the algebras being obtained by "equating the central element to a constant."

The algebra $\mathrm{Fun}(G^*)$ is obtained by reduction from the algebra of functions on the double D_+. Similarly, the algebra $\mathrm{Fun}(G^*)_c$ (i.e., the Poisson algebra of functions on the "section" $G^* \times \{c\} \subset \hat{G}^*$) can be obtained by reduction from the algebra of functions on the *twisted double.*

It is remarkable that the definition of the twisted double, and also the description of the Poisson structure on $\mathrm{Fun}(G^*)_c$, contains, not the differentiation ∂, but the finite automorphism $\exp(c\partial) \in \mathrm{Aut}(\mathfrak{G}, \mathfrak{G}^*)$. Therefore, we can define the twisted double and twisted bracket on G^* in a more general situation.

Let $D = G \times G$, $\tau \in \mathrm{Aut}\, G$ be an automorphism. We assume that the corresponding automorphism of the Lie algebra commutes with the r matrix and preserves the scalar product in $\mathfrak{G}$. Let

$$\hat{\tau} = \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix} \in \mathrm{Aut}(G \times G).$$

We set

$$^{\tau}r_d = \hat{\tau} r_d \hat{\tau}^{-1} = \begin{pmatrix} r & -2r_+ \circ \tau^{-1} \\ 2\tau \circ r_- & -r \end{pmatrix} \tag{5.6}$$

and define the Poisson bracket on D by

$$2\{\varphi, \psi\}_\tau = \langle\langle r_\partial X, Y \rangle\rangle + \langle\langle {}^{\tau} r_\partial X', Y' \rangle\rangle, \tag{5.7}$$

where X, X', Y, Y' are the right and left gradients of φ and ψ defined by (2.5). [Note that the Jacobi identity for (5.7) holds by virtue of the Yang—Baxter identity for r_∂, ${}^{\tau}r_\partial$.)

The group D equipped with the bracket (5.7) is called the *twisted double*; we denote it by D_τ. As in Sec. 2, we assume that G is a matrix algebraic group. Then the ring of functions Fun(D) is generated by the matrix elements of matrices $\rho(x)$ and $\rho(y)$, where ρ is a faithful matrix representation of G. The Poisson brackets between the generators are given by

$$\{x_1, x_2\}_\tau = \tfrac{1}{2}(r x_1 x_2 + x_1 x_2 r), \quad \{y_1, y_2\}_\tau = \tfrac{1}{2}(r y_1 y_2 + y_1 y_2 r),$$

$$\{y_1, x_2\}_\tau = -r_+ y_1 x_2 - y_1 x_2 r_+^\tau, \quad r_+^\tau = (\mathrm{id} \otimes \tau) r_+ \in \mathfrak{G} \otimes \mathfrak{G}. \tag{5.8}$$

Let G and G^* be dual Lie—Poisson groups.

Proposition 5.4. **1**. *The actions of the group G on D_τ by left and right shifts in accordance with the formulas*

$$G \times D \to D : g(x, y) = (gx, gy), \qquad D \times G \to D : (x, y)g = (xg, yg^\tau)$$

are admissible and form a dual pair.

2. *Similarly, the actions of the group G^* on D_τ in accordance with the formulas*

$$G^* \times D \to D : h(x, y) = (h_+ x, h_- y), \qquad D \times G^* \to D : (x, y)h = (xh_+, yh_-^\tau)$$

are also admissible and form a dual pair.

The reduced spaces D/G, $G\backslash D$ can be identified with the group G by means of the projections

$$p : D \to G : (x, y) \mapsto y^{-1}x, \quad p' : D \to G : (x, y) \mapsto x(y^{-1})^{\bar{\tau}}$$

(where for brevity we denote $\bar{\tau} = \tau^{-1}$). The reduced bracket on G is defined on the generators of its affine ring by

$$\{L_1, L_2\}_\tau = L_1 r_+^\tau L_2 + L_2 r_-^\tau L_1 - \tfrac{1}{2} r L_1 L_2 - \tfrac{1}{2} L_1 L_2 r, \tag{5.9}$$

where $r_+^\tau = (\mathrm{id} \otimes \tau) r_+$, $r_-^\tau = (\tau \otimes \mathrm{id}) r_-$ [note that $(\tau \otimes \tau) r = r$ by virtue of the invariance of r].

Another description of the reduced spaces D/G, D/G^* can be obtained by means of the factorization problem on D. Consider the decomposition

$$(x, y) = (L^+, L^-)(T, T^\tau)^{-1}, \quad (L^+, L^-) \in G^* \subset D, \quad T \in G. \tag{5.10}$$

Such a decomposition exists for almost all x, y. Thus, up to submanifolds of positive codimension we can identify $G^*\backslash D \simeq G$, $D/G \simeq G^*$ and define reduced brackets on the generators of the affine rings G, G^*. The reduced bracket on $G^*\backslash D \simeq G$ is the ordinary Sklyanin bracket,

$$\{T_1, T_2\} = \{r, T_1 T_2\},$$

and it does not depend on the twisting. The bracket of the generators L^+, L^- is given by

$$\{L_1^\pm, L_2^\pm\}_\tau = \tfrac{1}{2}[r, L_1^\pm L_2^\pm], \qquad \{L_1^+, L_2^-\}_\tau = r_+ L_1^+ L_2^- - L_1^+ L_2^- r_+^\tau. \tag{5.11}$$

The connection with formula (5.8) is given by means of the twisted factorization problem:

$$L = L_+^{-1} L_-^\tau.$$

Finally, using the decomposition (5.10), we can also express the bracket on D_τ in terms of the generators $L^\pm$, T. It is sufficient to calculate only the bracket $\{L_1^\pm, T_2\}$. The calculation gives

$$\{L_1^\pm, T_2\} = L_1^\pm T_2 r_\pm,$$

i.e., the same formula as in the untwisted case.

Now suppose $\tau = \exp(c\partial)$, where ∂ is a differentiation of $(\mathfrak{G}, \mathfrak{G}^*)$ that satisfies the condition (5.3). Then the bracket (5.10) is identical to the bracket on $\mathrm{Fun}(G^*)_c$ constructed from the central extension of the bialgebra $(\mathfrak{G}, \mathfrak{G}^*)$ by means of the cocycle (5.1).

The appearance of finite automorphisms instead of differentiations in formulas (5.8)—(5.11) is a manifestation of a general principle: For quantum groups (and even for Poisson Lie groups) the difference analogs appear in place of the differential operators. In the given case, the deformations of the Poisson structures are parametrized by means of the group of outer automorphisms $\mathrm{Out}(G, G^*)$, which, generally speaking, is larger than the "infinitesimal" group $[\mathrm{Der}(\mathfrak{G}, \mathfrak{G}^*)] \simeq H^2(\mathfrak{G})$.

A typical example when the group $\mathrm{Out}(\mathfrak{G}, \mathfrak{G}^*)$ is fairly large is associated with lattice systems. We describe it in more detail.

Let $(\mathfrak{G}, \mathfrak{G}^*)$ be a Lie bialgebra, $\mathcal{G} = \overset{N}{\oplus}\mathfrak{G}$, $\mathcal{G}^* = \overset{N}{\oplus}\mathfrak{G}^*$. It is convenient to regard elements of $\mathcal{G}, \mathcal{G}^*$ as functions on $\mathbb{Z}/N\mathbb{Z}$ with values in $\mathfrak{G}$, $\mathfrak{G}^*$. Let $\tau \in \mathrm{Out}\,\mathcal{G}$ be a cyclic permutation. Clearly, τ is an automorphism of the bialgebra $(\mathcal{G}, \mathcal{G}^*)$. In the "continuum limit" the periodic chain $\mathbb{Z}/N\mathbb{Z}$ becomes the circle, the algebra $\mathcal{G}$ becomes the loop algebra $L\mathfrak{G}$, and the automorphism τ becomes an outer differentiation ∂_x of the algebra $L\mathfrak{G}$, associated with its canonical central extension. The twisting by means of the automorphism τ of the Poisson algebras $\mathrm{Fun}(G^N)$ and their corresponding quantum algebras imitates in the finite-dimensional situation the effects of the central extension of the loop algebra.

The affine ring of the group G^N is generated by the matrix elements $\rho(L^s)_{ij}$, $s \in \mathbb{Z}/N\mathbb{Z}$. Specializing formulas (5.9) in the case when τ is a cyclic permutation, we obtain the Poisson brackets

$$\{L_1^k, L_2^k\}_\tau = -\tfrac{1}{2} r L_1^k L_2^k - \tfrac{1}{2} L_1^k L_2^k r, \quad \{L_1^k, L_2^{k+1}\}_\tau = L_1^k r_+ L_2^{k+1}, \quad \{L_1^k, L_2^l\}_\tau = 0 \quad \text{for } (k-l) \geq 2. \tag{5.12}$$

The bracket (5.12) is *nonultralocal*, i.e., does not decompose into a direct product of brackets on the factors. The properties of this bracket are described by the following theorem.

THEOREM 5.5. **1.** *Let* M: $G^N \to G$:$(L^1, \ldots, L^N) \to L^n \ldots L^2 L^1$ *be a monodromy mapping. If* G^N *is equipped with the bracket* (3.11), *and* G *with the bracket* (2.13), *then* M *is a Poisson mapping.* **2.** *Suppose that* N *is odd. Then the ring of Casimir functions of bracket* (3.11) *is generated by the functions*

$$C_k(L^1, \ldots, L^N) = \mathrm{tr}\, M^k, \quad k = 1, 2, \ldots, \tag{5.13}$$

and its symplectic sheets are orbits of the gauge transformations

$$G^N \times G^N \to G^N : (g, L) \mapsto \left(g^1 L^1 (g^2)^{-1}, \ldots, g^N L^N (g^1)^{-1}\right).$$

Remark. The property of the monodromy matrix described in Theorem 5.5 plays an important role in the theory of integrable systems. More generally, let $R \in \mathrm{End}(\mathfrak{G} \oplus \mathfrak{G})$ be an arbitrary solution of the modified Yang—Baxter equation on the square $\mathfrak{G}$,

$$R = \begin{pmatrix} A & B \\ B^t & D \end{pmatrix}, \qquad A = -A^t, \quad D = -D^t.$$

We define a Poisson bracket on G by setting

$$\{L_1, L_2\}_R = A L_1 L_2 - L_1 L_2 D + L_1 B L_2 - L_2 B^t L_1. \tag{5.14}$$

The bracket $\{,\}_R$ admits twisting by means of cyclic permutation on the lattice:

$$\{L_1^k, L_2^k\}_{R,\tau} = A L_1^k L_2^k - L_1^k L_2^k D, \quad \{L_1^k, L_2^{k+1}\}_{R,\tau} = L_1^k B L_2^{k+1}, \quad \{L_1^k, L_2^l\}_{R,\tau} = 0, \quad |k-l| \geq 2. \tag{5.15}$$

We assume in addition that $A - B^t = D - B$. Then the monodromy mapping

$$M : G^N_{(R,\tau)} \to G_{(R)}$$

is a Poisson mapping. The functions (5.13) are in involution with respect to the bracket (5.15).

Note also that for semisimple and affine Lie algebras the solutions of the Yang—Baxter equation (2.8) on the square $\mathfrak{G}$ can be completely classified (cf. [18]); r, the matrix of the double (22), is a special case of such a solution.

We now turn to the description of the *quantum twisted double* $\mathrm{Fun}_q(D_\tau)$. We assume that τ is an automorphism of the algebra End V such that $(\tau\otimes\tau)R^{VV}=R^{VV}$. We set

$$R^{\tau} = (\mathrm{id}\otimes\tau)R. \tag{5.16}$$

We denote by $\mathrm{Fun}_q({}^{\tau}G)$, $\mathrm{Fun}_q({}^{\tau}G^*)$ the algebras generated by the usual generators T, $L^{\pm}$ and by the relations (3.6)—(3.8), in which the R matrix R is replaced by R^{τ}. We define the algebra $\mathcal{A}_{\tau} = \mathrm{Fun}_q(D_{\tau})$ as the free algebra whose generator is given by the matrix elements of the matrices X, $Y\in\mathcal{A}_{\tau}\otimes\mathrm{End}\,V$, which satisfy the relations

$$X_2X_1 = R_+X_1X_2R_-, \quad Y_2Y_1 = R_+Y_1Y_2R_-, \quad Y_2X_1 = R_+X_1Y_2R_+^{\tau}. \tag{5.17}$$

It is obvious that the algebra with the relations (5.17) is a quantization of the Poisson algebra $\mathrm{Fun}(D_{\tau})$.

The analog of the Poisson reduction described in Proposition 5.4 is the following construction, which uses the structure of the dual comodule of the algebra $\mathrm{Fun}_q(D_{\tau})$.

Proposition 5.6. 1. *The formulas*

$$\Delta_LX = T\dot{\otimes}X, \quad \Delta_LY = T\dot{\otimes}Y, \quad \Delta_RX = X\dot{\otimes}T^{-1}, \quad \Delta_RY = Y\dot{\otimes}(T^{\tau})^{-1}, \quad T^{\tau} = (\mathrm{id}\otimes\tau)T \tag{5.18}$$

define on $\mathrm{Fun}_q(D_{\tau})$ *the structure of a left* $\mathrm{Fun}_q(G)$ *and right* $\mathrm{Fun}_q(G)$ *comodule.* **2.** *The subalgebra of "left-invariant functions," i.e., elements* $f\in\mathrm{Fun}_q(D_{\tau})$ *that satisfy the condition* $\Delta_Lf\in 1\otimes\mathrm{Fun}_q(D_{\tau})$, *is generated by the matrix elements of the matrix* $L=Y^{-1}X$; *the commutation relations L have the form*

$$L_2R_+L_1(R_+^{\tau})^{-1} = R_-^{\tau}L_1R_-^{-1}L_2.$$

Similarly, the subalgebra of right-invariant functions in $\mathrm{Fun}_q(D_{\tau})$ *is generated by the matrix elements of the matrix* $'L=X(Y^{\tau})^{-1}$; *we have*

$$R_+^{-1\prime}L_2R_+^{\tau\prime}L_1 = {}'L_1(R_-^{\tau})^{-1\prime}L_2R_-.$$

In addition $L_1'L_2={}'L_2L_1$, *i.e., the subalgebra of right- and left-invariant functions centralize each other.*

Formulas (5.17) are the quantum analog of formulas (5.8).

In the algebra $\mathrm{Fun}_q(D_{\tau})$ we can specify a different system of generators analogous to the generators $L^{\pm}$, T.

Proposition 5.7. *The algebra whose generators are the matrix elements of the matrices* $L^{\pm}$, T *satisfying the relations*

$$L_2^{\pm}L_1^{\pm} = R_+L_1^{\pm}L_2^{\pm}R_+^{-1}, \quad T_2T_1 = R_+T_1T_2R_+^{-1}, \quad L_2^{-}L_1^{+} = R_+L_1^{+}L_2^{-}(R_+^{\tau})^{-1}, \quad L_1^{\pm}T_2+ = T_2L_1^{\pm}R_{\pm}, \tag{5.19}$$

is isomorphic to the algebra $\mathrm{Fun}_q(D_+)$. *The correspondences between the generators X, Y and* $L^{\pm}$, *T are given by* $X=L_+T^{-1}$, $Y=L^{-}(T^{\tau})^{-1}$, *where* $T^{\tau}=(\mathrm{id}\otimes\tau)T$.

The main example of twisting that we consider is associated with lattice systems. Suppose, as in Theorem 5.5, the group G^N is regarded as a group of functions on the periodic lattice $\mathbb{Z}/N\mathbb{Z}$ with values in G. The quantum algebra $\mathrm{Fun}_q(G^N)$ can be described by means of the generators T^i, $i\in\mathbb{Z}/N\mathbb{Z}$, and the relations

$$R_+T_1^iT_2^i = T_2^iT_1^iR_+, \quad T_1^iT_2^j = T_2^jT_1^i, \quad i\neq j. \tag{5.20}$$

Let

$$R^{i-j} = \begin{cases} R, & i=j,\\ I, & i\neq j.\end{cases} \tag{5.21}$$

The matrix R^{i-j} can be regarded as an operator in $\overset{N}{\otimes}V$. Let τ be a cyclic permutation, $\tau(i)=(i+1)(\mathrm{mod}\,N)$. Then $R^{\tau i-\tau j}=R^{i-j}$, and the automorphism τ makes it possible to twist the quantum algebras of functions. We arrive at the following algebras associated with the lattice:

1. The twisted double with generators X^i, Y^i, $i\in\mathbb{Z}/N\mathbb{Z}$, and relations

$$X_2^jX_1^i = R_+^{i-j}X_1^iX_2^jR_-^{i-j}, \quad Y_2^jY_1^i = R_+^{i-j}Y_1^iY_2^jR_-^{i-j}, \quad Y_2^jX_1^i = R_+^{i-j}X_1^iY_2^jR_+^{i-j-1}. \tag{5.22}$$

2. The twisted algebra $\mathrm{Fun}_q(G_{\tau}^*)\simeq\mathrm{Fun}_q(D_{\tau}/G)$ with generators L^i and relations

$$L_2^jR_+^{i-j}L_1^i(R_+^{i-j-1})^{-1} = R_-^{i-j+1}L_1(R_-^{i-j})^{-1}L_2^j. \tag{5.23}$$

THEOREM 5.8 }8]. **1.** *Let*

$$M = \overset{\curvearrowleft}{\prod} L^i$$

be the monodromy matrix; then

$$M_2 R_+ M_1 R_+^{-1} = R_- M_1 R_-^{-1} M_2. \quad (5.24)$$

Thus, the monodromy generates the embedding

$$\mathrm{Fun}_q(G^*) \hookrightarrow \mathrm{Fun}_q(D_\tau/G).$$

2. *Let N be odd. Then the algebras* $\mathrm{Fun}_q(D_\tau/G)$ *and* $\mathrm{Fun}_q(G^*) \subset \mathrm{Fun}_q(D_\tau/G)$ *have the same centers.*

We have described above the structure of the center of the algebra $\mathrm{Fun}_q(G^*)$ in Proposition 3.10.

The algebra $\mathrm{Fun}_q(D_\tau/G)$ is a deformation of the algebra $U_q(\mathfrak{G})^{\otimes N}$. It can be shown that in the "continuum limit" this algebra goes over into the algebra $U(L\mathfrak{G})_c = U(\widehat{L\mathfrak{G}})/(c = \mathrm{const})$, i.e., into the deformation of the universal covering loop algebra $L\mathfrak{G}$ obtained by "equating the central charge to a constant."

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