Higher genus associators and quantum groups

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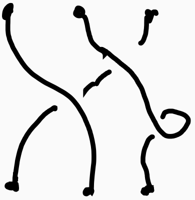
The braid category

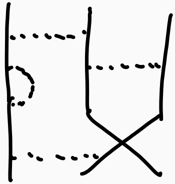
- Fix **k** field of characteristic 0, and let fB be the category having
  - objects finite sequences of •
  - morphisms formal **k**-linear combination of framed braids



- Composition is given by stacking, tensor product by juxtaposing.
- This is the free/universal (balanced) braided monoidal category.
- It means every M in a balanced braided monoidal category $\mathcal{A}$ induces a braided functor $\mathrm{fB} \to \mathcal{A}$.

The diagram category

- Let CD be the category having
  - objects finite sequences of •
  - morphisms formal linear combinations of *horizontal chord diagrams*[1]



  modulo a certain set of quadratic relations.
- Composition is given by stacking, tensor product by juxtaposing.
- This is the free/universal *infinitesimal* braided monoidal category.

[1] Really the degree completion thereof, the grading is given by the number of chords.

Associators

- Roughly speaking, an infinitesimal braided monoidal category is a symmetric monoidal category with symmetry P equipped with a natural endomorphism t of $- \otimes -$ such that

$$(\mathrm{Id} + \hbar t/2) \circ P$$

satisfies the braiding axioms modulo $\hbar^2$.

- The standard example is $(\mathrm{Rep}\, G, t)$ where G is a reductive algebraic group over $\mathbf{k}$ and t is the canonical element associated with the Killing form.

Definition

A lift of the infinitesimal braided structure on CD *to an actual braided structure is determined by an element* $\Phi \in \mathrm{CD}(\bullet \bullet \bullet)$ *satisfying a set of algebraic equations. Such an element is called a Drinfeld associator.*

Theorem (Drinfeld)

The set of associators is non-empty.

- The proof of this for $\mathbf{k} = \mathbb{C}$ is analytic and uses the monodromy of the so-called KZ equation in conformal field theory. He then deduces this is true for any $\mathbf{k}$ but the proof is non-constructive.
- One gets this way a braided functor

$$\mathsf{fB} \longrightarrow \mathsf{CD}^{\Phi}$$

- This construction has a version for knots and tangles (combinatorial construction of the so-called Kontsevich integral of knots).
- This becomes an equivalence upon passing to a certain completion of fB: this gives a combinatorial model for the rational homotopy type of the little discs operad (Kontsevich-Tamarkin formality).

The Kohno–Drinfeld theorem

- This also plays a prominent role in deformation quantization.
- Let $\operatorname{Rep}_\Phi G$ be the trivial deformation $\operatorname{Rep} G[[\hbar]]$ as a category, with braided monoidal structure given by an associator Φ. Then:

Theorem (Drinfeld)

Let $\operatorname{Rep}_\hbar G$ *be the category of representation of the quantum group associated with* G*. Then there is a braided equivalence*

$$\operatorname{Rep}_\hbar G \simeq \operatorname{Rep}_\Phi G.$$

- This explains why quantum groups exist at all.
- This also recovers a theorem of Kohno saying the monodromy of the KZ equation can be computed explicitly using quantum groups.

Elliptic associators

- Calaque–Enriquez–Etingof: notion of elliptic associator, related to the KZB equation on elliptic curves.
- They give an explicit formula using an ordinary associator.
- This gives a universal representations of the category of braids in a topological torus into a category of so-called (ordered) symplectic chord diagrams.
- Just like ordinary chords diagrams are diagrammatic versions of $(U(\mathfrak{g})^{\otimes n})^{\mathfrak{g}}$, those are diagrammatic versions of $(D(\mathfrak{g}) \otimes U(\mathfrak{g})^{\otimes n})^{\mathfrak{g}}$ where $D(\mathfrak{g})$ is the algebra of differential operators on $\mathfrak{g}$, a.k.a. the Weyl algebra on the underlying vector space.
- Observe that $D(\mathfrak{g})$ is a quantization of $T^*\mathfrak{g}$.

Outline

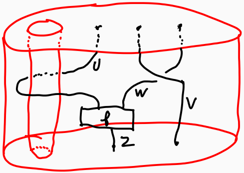
- Goal: understand elliptic associator, compute explicitly using well-known quantum algebras, generalize to higher genus.
- To any braided monoidal category there is a TFT-like "skein category" construction for any surface which carries representations of the associated braid group.
- One might want to compute the skein category of CD^{Φ} for the punctured torus $\mathbb{T}^*$. This gives something complicated, roughly because it is related to a quantization of $(G \times G)/G$, the character variety of $\mathbb{T}^*$, rather than $T^*\mathfrak{g}/G$.
- This has a "trigonometric" degeneration (corresponding to a quantization of T^*G/G) leading to a certain refinement of elliptic associators. The version for the closed torus $\mathbb{T}$ is, however, again complicated.
- This has a further "rational" degeneration associated with $T^*\mathfrak{g}$, which recover elliptic associator for $\mathbb{T}$.
- Using previous work joint with Ben-Zvi–Jordan we can compute specializations of all of those using quantum groups.

Character varieties

- Let S be a surface and let
$$\mathrm{CH}(S) = \{\rho : \pi_1(S) \to G\}/G$$
be the *character variety* of S.
- A classical result by Atiyah–Bott and Goldman states these have a canonical Poisson structure.
- A crucial observation is that for the disc $\mathbb{D}$, in a appropriate stacky sense,
$$\mathrm{CH}(\mathbb{D}) = BG := pt/G$$
so that the category of sheaves
$$\mathrm{QC}(\mathrm{CH}(\mathbb{D})) \simeq \mathrm{Rep}\, G.$$
- $\mathrm{Rep}_\Phi\, G$ is a quantization of a certain (shifted) Poisson structure on BG, of which CD^Φ is a diagrammatic version.
- We also have ($\mathbb{A}$ is the annulus)
$$\mathrm{CH}(\mathbb{A}) = G/G \qquad \mathrm{CH}(\mathbb{T}^*) = (G \times G)/G.$$

Skein categories

- If $\mathcal{A}$ is any **k**-linear (balanced) braided monoidal category, the skein category $\int_S \mathcal{A}$ has
  - objects collections of points on S
  - morphisms **k**-linear combinations of isotopy classes of certain framed graphs embedded in $S \times I$, with edges labelled by objects of $\mathcal{A}$ and vertices labelled by morphisms in $\mathcal{A}$.
  - Every chunk of such a graph inside an embedded $\mathbb{D} \times I$ inside $S \times I$ can be seen as a morphism in $\mathcal{A}$, and roughly speaking we mod out by the relations that are satisfied in $\mathcal{A}$.



- For every $V \in \mathcal{A}$, $M \in \int_S \mathcal{A}$: action of $B_n(S)$ on $M \otimes V^{\otimes n}$.

Universal skein category

- This is due to Walker and is closely related to factorization homology (Johnson–Freyd, Cooke).
- We have $\int_{\mathbb{D}} \mathcal{A} \simeq \mathcal{A}$, and the assignment $S \mapsto \int_S \mathcal{A}$ is a TFT in a very strong sense, i.e. there is a simple algebraic rule to compute it by cutting a surface into elementary pieces.
- Tautologically, $\int_S \mathrm{fB}$ is $\mathrm{fB}(S)$, the category of framed braids in $S \times I$. In that particular case, this TFT-like behaviour was discovered by Yetter.
- Long story short we get for free functors

$$\mathrm{fB}(S) \longrightarrow \int_S \mathrm{CD}^{\Phi} .$$

- So we want to compute the RHS, but there's a catch !

The catch

- There is s straightforward notion of chord diagrams in $S \times I$, equivalently of chord diagrams labelled by $\pi_1(S)$.

Theorem

If S has a non-empty boundary, the hom spaces in $\int_S \mathrm{CD}^\Phi$ are isomorphic as vector spaces to the corresponding space of diagrams in S.

- ... but the composition in this category depends non-trivially on Φ. This is essentially a reformulation of a result by Andersen–Mathes–Reshtikhin.
- In fact they observe the algebra of *closed* diagrams on S is a commutative, in fact Poisson, algebra, a diagrammatic version of $\mathcal{O}(\mathrm{CH}(S))$.
- I don't know what happens for closed surfaces.

Explanation

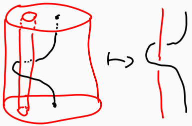
- Informally, that $\mathrm{CD}^{\Phi} = \mathrm{CD}$ as category mirrors the fact that the Poisson structure on $\mathrm{CH}(\mathbb{D})$ is trivial (but it has a non-trivial *shifted* Poisson structure which is responsible for the non-trivial braiding).
- Likewise, $\int_S \mathrm{CD}^{\Phi}$ (really, the version for tangles) is a universal quantization of $\mathrm{CH}(S)$ which has a complicated and non-trivial Poisson structure.
- Roughly speaking, AMR result says this quantization is *flat* in the case of punctured surfaces.
- On the other hand, if S has genus g and $n \geq 1$ boundary components, one has a formal Poisson isomorphism

$$\left((T^*\mathfrak{g})^g \times (\mathfrak{g}^*)^{n-1}\right)/G \to \mathrm{CH}(S) = (G^{2g} \times G^{n-1})/G.$$

- There is a version for closed surfaces given by Hamiltonian reduction.
- This is a combinatorial version of the Riemann-Hilbert correspondence (Hitchin, Jeffreys, Alekseev–Malkin–Meinrenken, Naëf, ...)

The annulus

- One can regard $\mathsf{fB}(\mathbb{A})$ as the sub-category of fB of braids whose first strand is fixed and has trivial framing.



- Restricting the chord diagram representation for fB this induces (monoidal !) functors

$$\mathsf{fB}(\mathbb{A}) \longrightarrow \int_{\mathbb{A}} \mathsf{CD}^{\Phi} \longrightarrow \mathsf{CD}^{1}$$

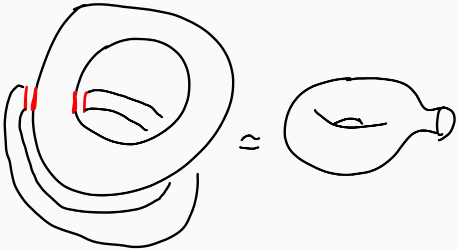
where CD^1 is the sub-category of CD with first strand fixed and with no self-chord.

- The functor $\int_{\mathbb{A}} \mathsf{CD}^{\Phi} \to \mathsf{CD}^1$ is a diagrammatic quantization of the formal Poisson map $\mathfrak{g}^*/G \to \mathsf{CH}(\mathbb{A}) = G/G$.

Traces

- Recall that if A is an algebra and M an A-bimodule, the trace of M is the vector space quotient $M/[A, M]$.
- This construction has a categorical counterpart: if $\mathcal{A}$ is a monoidal category, $\mathcal{M}$ an $\mathcal{A}$-bimodule, there is a categorical trace $\mathrm{Tr}_{\mathcal{A}}(\mathcal{M})$.
- The TFT formalism implies that $\mathrm{fB}(\mathbb{A})$ is a fB-bimodule, and that if $\mathbb{T}^*$ is a punctured torus

$$\mathrm{fB}(\mathbb{T}^*) \simeq \mathrm{Tr}_{\mathrm{fB}}(\mathrm{fB}(\mathbb{A})).$$



Degeneration

- Likewise, CD^1 is a CD-bimodule, and also a CD^{Φ} bimodule, where now of course the action depends on the associator.

Theorem

We have an equivalence of categories

$$\mathsf{Tr}_{\mathsf{CD}^{\Phi}}(\mathsf{CD}^1) \simeq \mathsf{Tr}_{\mathsf{CD}}(\mathsf{CD}^1).$$

- The RHS do not depends on Φ and has a fairly straightforward description as a category $\mathsf{CD}_1^{1,trig}$ of diagrams.
- By functoriality of traces we get a representation

$$\mathsf{fB}(\mathbb{T}^*) = \mathsf{Tr}_{\mathsf{fB}}(\mathsf{fB}(\mathbb{A})) \longrightarrow \mathsf{CD}_1^{1,trig}$$

given by an explicit formula involving Φ.

Specialization

- The skein construction leads to a specialization
$$\int_{\mathbb{T}^*} \mathrm{CD}^{\Phi} \rightsquigarrow \int_{\mathbb{T}^*} \mathrm{Rep}_{\Phi}\, G.$$
which, using previous work with Ben-Zvi–Jordan can be computed using an explicit algebra $D_{\hbar}(G)$ of quantum differential operators on G via the original KD equivalence.
- Further, one can show there is a compatible specialization
$$\mathrm{CD}_1^{1,trig} \rightsquigarrow \hat{D}(G)\text{-mod}_{\mathrm{Rep}\,\mathfrak{g}}[[h]]$$
where the RHS is the quantization of T^*G/G coming from diff. operators on G.

Theorem

The category $\hat{D}(G)\text{-mod}_{\mathrm{Rep}\,\mathfrak{g}}[[h]]$ carries an action of the punctured torus braid group which can be computed explicitly using $D_{\hbar}(G)$.

Closed torus and elliptic associators

- We then have a further degeneration $\mathsf{CD}_1^{1,\mathit{trig}} \to \mathsf{CD}_1^1$ where CD_1^1 is the category of ordered symplectic diagrams. This recover CEE's formula for an elliptic associator.
- It specializes to a map

$$\hat{D}(G)\text{-mod}_{\mathrm{Rep}\,\mathfrak{g}}[[\hbar]] \to \hat{D}(\mathfrak{g})\text{-mod}_{\mathrm{Rep}\,\mathfrak{g}}[[\hbar]]$$

  which is essentially the push-forward along $\exp : \hat{\mathfrak{g}} \to \hat{G}$.
- Roughly speaking, this extra degeneration is needed to handle the case of the closed torus. Also, CD_1^1 is graded while the other ones are only filtered.
- The TFT formalism gives a recipe for "closing the hole", by formally modding out by certain relations.
- In the case of CD_1^1 those can be rewritten in a way that do not involve the associator.
- This again can be specialized to a certain category of $D(\mathfrak{g})$-modules, which in turns can be computed explicitly using $D_\hbar(G)$.

Applications and outlook

- Hence we get an elliptic version of the Kohno–Drinfeld theorem, in particular an explicit computation of the monodromy of the elliptic KZB equation.
- There is a similar story in higher genus, leading to representations

  $$\mathrm{fB}(S_n^g) \to \mathrm{CD}_g^{n,trig} \to \mathrm{CD}_g^n$$

  for punctured surfaces, which also descend to closed surfaces. In particular one gets a generalization of the Kontsevich integral for any surface. This should be related to recent work of Habiro–Massuyeau.
- In principle one should be able to write down formulas for higher genus analogs of associators (in progress...).
- Along the way this shows the AMR quantization of character varieties coincides, upon specialization, to certain well-known combinatorial quantizations constructed by Alekseev.

Thank you for your attention !