What is. . . The Kontsevich integral ?

Adrien Brochier

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- a powerful knot invariant, taking values in a space of Feynman diagrams
- equivalent to the collection of all so-called finite type invariants
- strictly stronger that all "quantum" invariants
- defined from a perturbative expansion of the WZW conformal field theory
- deeply related to deformation-quantization.

Finite type invariants

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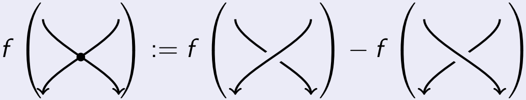
let f be a knot invariant. Its extension to the space of singular knots is defined using the skein relation:



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- This defines a filtration on the space of invariants.
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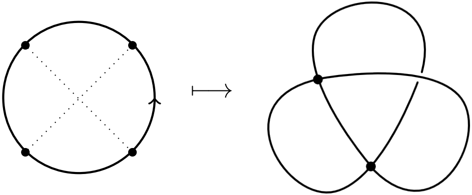
- More generally, is f is of degree n, then f applied to a knot with exactly n singularities is blind to the topology.
- Hence, it knows only the combinatorial information given by the position of the singularities, encoded by a *chord diagram*:



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Question

Can we go the other way around ? Can every invariant of diagram be integrated to a knot invariant ?

Chord diagrams as Feynman diagrams

- Chord diagrams are related to lie algebras: if $\mathfrak{g}$ is a Lie algebra equipped with an invariant, symmetric pairing (e.g. $\mathfrak{gl}_n$ with $(A, B) \mapsto \mathrm{Tr}(AB)$), let $t = \sum e_i \otimes e^i \in \mathfrak{g} \otimes \mathfrak{g}$ and V a finite dimensional $\mathfrak{g}$-module.

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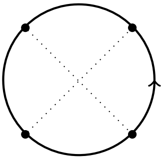
- Every chord diagram can be paired with this data to produce a number. This descends to the quotient by the 4T relations, and upon renormalization by the 1T relation as well.
- Roughly: each dotted line is a copy of t. Multiply its components in the correct order, act on V and take the trace.



$$\longmapsto \mathrm{Tr}_V\left(\sum e_i e_j e^i e^j\right)$$

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- This system is integrable, and equivariant w.r.t. the obvious action of S_n on C_n and $V^{\otimes n}$.
- By analytic continuation of solutions, one gets a representation of the braid group B_n on $V^{\otimes n}[[\hbar]]$.

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- Each term is a product of two number: the one obtained by pairing the diagram with $(\mathfrak{g}, t, V)$ which doesn't depends on the knot, and one obtained as a value of a certain integral along the knot, which doesn't depends on $\mathfrak{g}$.
- Hence, one can write down a universal version of this as a formal sum of diagrams (i.e. without having to choose $(\mathfrak{g}, t, V)$).

Kontsevich theorem

This is the Kontsevich integral:

$$Z(K) = \nu^{1-\frac{c}{2}} \sum_{n\geq 0} \frac{1}{(2\pi i)^n} \int_{t_{min}<t_1<\cdots<t_n<t_{max}} \sum_{P=\{(z_i,z_i')\}} (-1)^{\downarrow} \bigwedge_{i=1}^{n} \frac{dz_i - dz_i'}{z_i - z_i'} D_P$$

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- ν is certain renormalization factor, and c is the number of critical points (w.r.t. t) on K.

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Theorem (Kontsevich)

The series $Z(K)$ is a knot invariant. Let I_n be the space of invariants of degree at most n. Then Z induces a filtered, linear isomorphism from I_n to the dual of the space of chord diagrams with at most n chords.

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- In other words, this is a universal finite type invariant in the following sense: let f be a finite type invariant, $\tilde{f}$ the invariant of chord diagram it induces. Then:

$$f = \tilde{f} \circ Z + l.o.t.$$