What is. . . Higher algebra ?

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- Those structures have a close connection with topological field theories, they also organize nicely a variety of interesting and concrete algebraic constructions.
- If one drops the locally constant property and/or replace $\mathbb{R}^n$ by an arbitrary (structured) manifold X one gets the notion of factorization algebra over X, which is expected to provide a formalization of observables in QFT's. In particular, factorization algebras on Riemann surfaces are close cousins of vertex algebras.

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- Hence this also encompass up-to-homotopy versions of algebras where associativity becomes a structure rather than a property. Examples are monoidal categories (in Cat with natural isomorphisms) or A_∞-algebras (in chain complexes).
- More generally we can talk about E_1-algebras in any E_1/monoidal ∞-category: this is an instance of the so-called microcosm principle.

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- Similarly, if $(\mathcal{C}, x \in \mathcal{C})$ is a pointed ordinary or ∞-category, then its "based loop space"

$$\Omega_x \mathcal{C} := \mathrm{End}_{\mathcal{C}}(x)$$

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- Conversly, every (ordinary or up-to-homotopy) algebra arise this way from $(A\text{-mod}, A)$.

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- For familiar examples, in vector spaces E_n-algebras for $n \geq 2$ are just commutative algebras, while E_2-algebras in Cat are braided monoidal categories.

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- Fun fact: if A is an (ordinary) algebra such that the multiplication $m : A^{\otimes 2} \to A$ is an algebra morphism, then A is commutative, i.e. commutative algebras are $Alg(Alg(vect))$.

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  - The functor $Alg(vect) \to Cat$ given by $A \mapsto A$-mod is monoidal, hence send algebras to algebras, so if A is commutative, A-mod is a monoidal category.

E_2-algebras

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  - For the same reason as before, modules over an E_2-algebra is a monoidal category.

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- More generally, playing with the identitification $E_n = E_{n-k}$ in E_k one gets a whole zoo of various notions of modules, corresponding to "codimension k defects".
- Every E_n-algebra have a center which an E_{n+1}-algebra.

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