# Quantization of character varieties, topological field theories and the Riemann-Hilbert correspondence

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# Rough plan

- Various point of views on $\mathrm{Rep}_q\, G$
- Quantization of character varieties
- Quantum Riemann-Hilbert correspondence

# $\operatorname{Rep}_q G$ **from rep. theory**

- Fix $G$ a simple (or reductive) complex algebraic group and $q \in \mathbb{C}^\times$. For the purpose of this talk $q$ is generic and I will often assume $G = \mathrm{SL}_2(\mathbb{C})$.
- $\operatorname{Rep}_q G$: category of (integrable) modules over the quantum group associated with $G$. This is a $q$-deformation/quantization of $\operatorname{Rep} G$.
- As mere categories $\operatorname{Rep}_q G \simeq \operatorname{Rep} G$, but it has a non-trivial and interesting braided monoidal (in fact ribbon) structure.
- Hence one gets representations of braids groups, and links invariants (e.g. the Kauffmann bracket/Jones polynomial).

# $\text{Rep}_q\, G$ from skein theory

- In $\text{Rep}_q\, \text{SL}_2$, every map between tensor powers (of the same parity) of $V = \mathbb{C}^2$ is the image of some tangle (Schur–Weyl duality).
- In turn those are determined by the value of "cup" and "cap" (making $V$ self-dual) by repeated applications of Kauffmann's skein relations:


- Define the Temperley-Lieb category $TL_q$ as having objects finite sequences of points on a disk, and morphisms (framed unoriented in that case) tangles modulo the above relations.

**Theorem (Folklore, Tingley for the correct signs)**

*The assignment* $\bullet \mapsto V$ *realizes* $\text{Rep}_q\, \text{SL}_2$ *as the completion of* $TL_q$ *under direct sums and splittings of idemptotents.*

# $\mathrm{Rep}_q\, G$ from CFT (aka the Drinfeld–Kontsevich integral)

- $\mathrm{Rep}_{\mathrm{KZ}}\, G = \mathrm{Rep}\, G$ with the same tensor product.
- Braided tensor structure defined analytically by integrating the so-called Knizhnik–Zamolodchikov connection on the configuration space of points in $\mathbb{C}$.
- Transcendental and highly non-explicit, but geometric and conceptually clear.

**Theorem (Kohno, Drinfeld, Kazhdan–Lusztig)**

*Equivalence of braided tensor categories*

$$\mathrm{Rep}_{\mathrm{KZ}}\, G \simeq \mathrm{Rep}_q\, G.$$

# $\mathrm{Rep}_q\, G$ as a quantization

- Quasi-classical limit: set $q = 1 + \hbar$ and expand

$$= \hbar t + O(\hbar^2)$$

where $t \in \mathfrak{g}^{\otimes 2}$ is the canonical element w.r.t the Killing form.
- $(\mathrm{Rep}\, G, t)$ is an "infinitesimal braided monoidal category".
- Poisson geometric interpretation: the sym. mon. $\mathrm{Rep}\, G$ is naturally identified with quasi-coherent sheaves ($=$ categorified algebra of functions) on the classifying stack $BG = pt/G$.
- $t$ turns $BG$ into a (2-shifted) symplectic stack in the sense of Pantev–Toën–Vaquie–Vezzosi.
- $\mathrm{Rep}_{\mathrm{KZ}}\, G$ is the quantization of $BG$ coming from the formality of the little disks operad.

# Character varieties

- Let $S$ be a topological surface. The representation variety of $S$ is

$$\mathsf{R}(S) := \{\pi_1(S) \to G\}$$

- The character variety of $S$ is

$$\mathsf{CH}(S) := \mathsf{R}(S)/G^{ad}.$$

**Theorem (Atiyah–Bott, Goldman, PTVV)**
$\mathsf{CH}(S)$ *carries a canonical Poisson structure.*

- Canonical $\longrightarrow$ diffeomorphism invariant.
- Really depends on the surface, not just on its $\pi_1$.

# Character varieties

- Observe that $\mathrm{CH}(D^2) = pt/G = BG$.
- Hence QC of $\mathrm{CH}(S)$ obtained by gluing together copies of Rep $G$.
- This is very non-true for the algebra of global functions (i.e. for the underlying variety as opposed to stack).

**Theorem (Ben-Zvi–Francis–Nadler)**

*The assignment $S \longmapsto QC(\mathrm{CH}(S))$ is part of a 2d topological field theory.*

- The Poisson structure is given by "integrating the Killing form".
- Fairly simple recipe for its behaviour under gluing.

**Slogan (Alekseev–Malkin–Meinrenken, Calaque, Safronov)**

Gluing $\longleftrightarrow$ (multiplicative) Hamiltonian reduction

# Examples and building blocks

- Assume $S$ has genus $g$ and $n > 1$ boundary components.
- Then $\pi_1(S) = F_{2g+n-1}$ is free, hence one has a (non-canonical !) identification
$$R(S) \cong G^{2g+n-1}.$$
- It implies that
$$QC(\mathrm{CH}(S)) \simeq \mathcal{O}(R(S))\text{-mod}_{\mathrm{Rep}\, G}\,.$$
- Fock-Rosly: the Poisson structure on $\mathrm{CH}(S)$ has a (again non-canonical) lift to $R(S)$.
- $R(\mathrm{Annulus}) \cong G_{STS}$, Poisson structure discovered by Semenov-Tian-Shansky.
- $R(T^2 \backslash pt) \cong G \times G$ the "Heisenberg double", a multiplicative version of $T^*G$.
- Other punctured and closed surfaces obtained by combining those two and using Hamiltonian reduction.

# Combinatorial quantization

- $G_{STS}$ quantizes to $\mathcal{O}_q(G)$, a well-known algebra which goes by various names (braided dual, reflection equation algebra,...). Discovered by Majid and Donin–Mudrov.
- The Heisenberg double quantizes to $D_q(G)$, a simultaneous $q$-deformation of $D(\mathfrak{g})$ and $D(G)$, which you might remember from Sam's talk this morning. This is $\mathcal{O}_q(G)^{\otimes 2}$ as an object, plus certain cross relations.
- Arbitrary punctured surfaces $S_{g,n}$: Alekseev's moduli algebras

$$D_q(G)^{\tilde{\otimes} g} \tilde{\otimes} \mathcal{O}_q(G)^{\tilde{\otimes} n-1}$$

(braided tensor product).
- Taking invariants leads to a quantization of $\mathcal{O}(\mathrm{CH}(S))$.

✓ Explicit and algebraic

× Depends on a combinatorial presentation of $S$. Topological features have to be proved by hand.

× Originally not defined for closed surfaces.

# Skein quantization

- If $M$ is a 3-fold, the skein module $\mathrm{Sk}(M)$ is the vector space of isotopy classes of links in $M$ modulo the skein relation.
- Functorial w.r.t to embeddings. In particular, if $S$ is a surface, $\mathrm{Sk}(S) := \mathrm{Sk}(S \times [0,1])$ is an algebra.
- $\mathrm{Sk}(S)$ is a quantization of $\mathrm{CH}(S)$ (Bullock, Frohman, Przytycki, Turaev, ...)

✓ Elementary to define, manifestly topological.

× Difficult to handle algebraically (already for $\mathrm{SL}_2$).

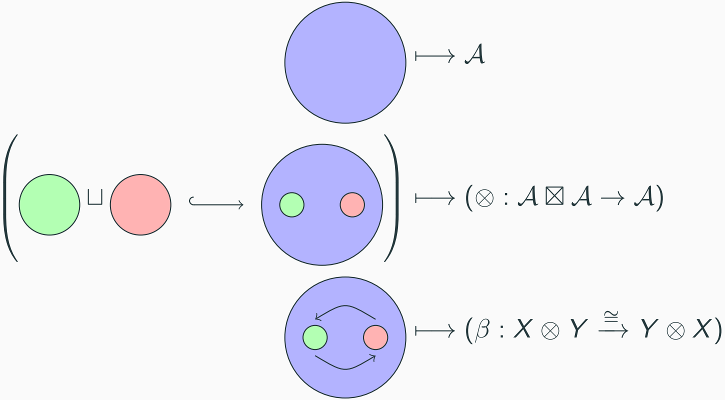
× No TFT/gluing formula around.

# Factorization homology

- Want the best of both worlds and more: a purely topological, TFT like construction, together with an explicit combinatorial description.
- Idea: use factorization homology to quantize categories of sheaves on $\mathrm{CH}(S)$.
- Let $\mathrm{Mfld}_2$ be the category with:
  - objects oriented top. surfaces
  - morphisms smooth embeddings
  - 2-morphisms isotopies between those
  - (and so on although we won't need it).
- Sym. mon. structure given by disjoint union.
- $\mathrm{Disk}_2$ is the full subcategory of finite disjoint unions of disks.

# Factorization homology

- A ribbon category $\mathcal{A}$ determines a sym. mon. functor $\mathrm{Disk}_2 \longrightarrow Cat_{\mathbb{C}}^{\boxtimes}$, a.k.a. an algebra over the little 2-disks/$E_2$ operad



**Theorem (Ayala–Francis–Tanaka)**

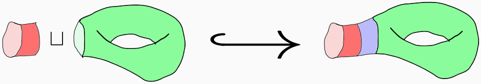
*There exists a canonical extension*

$$\begin{array}{ccc} \mathrm{Mfld}_2 & & \\ \uparrow & \searrow^{\int_{(-)} \mathcal{A}} & \\ \mathrm{Disk}_2 & \xrightarrow[\mathcal{A}]{} & \mathit{Cat}_{\mathbb{C}}^{\boxtimes} \end{array}$$

*which satisfies and is characterized by excision.*

# Factorization homology

- If $S = P \times [0,1]$, by functoriality w.r.t embedding $\int_S \mathcal{A}$ is an algebra (i.e. a monoidal category).
- If $S$ has boundary $P$, $\int_S \mathcal{A}$ is a module over it.



$$\int_{S^1\times[0,1]} \mathcal{A} \boxtimes \int_{T^2\setminus pt} \mathcal{A} \longrightarrow \int_{T^2\setminus pt} \mathcal{A}$$

**Excision**

$$\int_{X \underset{P\times[0,1]}{\sqcup} Y} \mathcal{A} \simeq \int_X \mathcal{A} \underset{\int_{P\times[0,1]} \mathcal{A}}{\boxtimes} \int_Y \mathcal{A}$$

# Main result

**Theorem (Ben-Zvi–B–Jordan)**

*Any choice of a combinatorial decomposition of a punctured surface $S$ induces an equivalence*

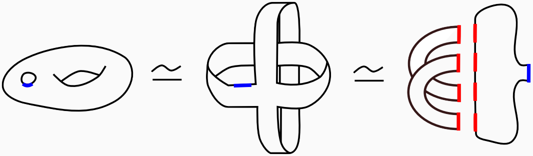
$$\int_{\mathrm{Rep}_q G} S \simeq A_S\text{-mod}_{\mathrm{Rep}_q G}$$

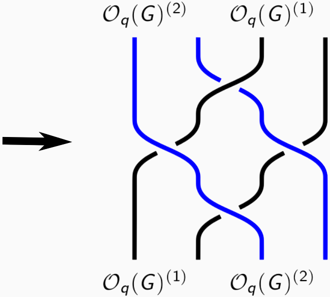
*where $A_S$ is the moduli algebra attached to the same decomposition.*

- We get those algebras as an output of a topological machine.
- Similar albeit slightly more complicated picture for closed surfaces and surfaces with marked points.
- For the closed torus with a suitable marking we get a certain category of equivariant $D_q(G)$-modules...
- ...and upon taking global sections category of modules for the sDAHA (with both parameters and its $\mathrm{SL}_2(\mathbb{Z})$-action).

# Braids and doughnuts are worth a thousand words

- One copy of $O_q(G)$ for each "flat handle", cross-relations dictated by the way those are attached to the disk on the right.





# Skein version

**Theorem (Cooke, Johnson-Freyd, after Walker)**

*$\int_{\mathrm{Rep}_q G} S$ is identified with an appropriate completion of the skein category of $S$, having objects finite collections of points on $S$, and morphisms tangles in $S \times [0,1]$ modulo the skein relation.*

- Endomorphisms of the empty collection is the skein algebra. Hence, this identifies the skein and the combinatorial quantization of global functions on $\mathrm{CH}(S)$.
- It shows that, in sharp contrast with skein algebras, skein categories satisfies excision.

# 3D extension

- If $M$ is a 3-fold, $\mathrm{CH}(M) \to \mathrm{CH}(\partial M)$ is Lagrangian.
- Gluing $\leftrightarrow$ Lagrangian intersection.
- With Jordan–Snyder we extend the previous construction to a 3d TFT (more on that on Thursday).
- This recovers skein modules and their relative versions.
- Of particular interest is the quantization of

$$\mathrm{CH}(S^3 \backslash K) \longrightarrow \mathrm{CH}(T^2)$$

for a knot $K$ (quantum A-polynomial).

# Riemann-Hilbert

- Let $C$ be a Riemann surface and $P$ a collection of $n > 0$ points on it.
- Let $\mathsf{M}(C, P)$ be the moduli space of meromorphic connections with regular singularities at each $x \in P$.
- The RH correspondence is an analytic isomorphism

$$\mathsf{M}(C, P) \cong \mathsf{CH}(S)$$

  taking a connection to its monodromy representation, where $S$ is the underlying surface of $C$ with a disk removed around each $x \in P$.
- This induces an iso between the formal neighborhood of the trivial connection and the trivial representation respectively.
- The former can be identified with $\widehat{\mathfrak{g}}^{2g+n-1}/G$.

# Riemann-Hilbert

- The $n$th braid group of $C$ acts on $\mathrm{M}(C, P)$ by isomonodromy, i.e. by moving the marked points while keeping the monodromy constant, and on $\mathrm{CH}(S)$ by moving the boundary components.
- Recall that $\mathfrak{g}^*$ has a linear Poisson structure coming from the Lie bracket.
- The Killing form gives $G$-equivariant identifications

$$\mathfrak{g} \cong \mathfrak{g}^* \qquad\qquad T^*\mathfrak{g} \cong \mathfrak{g} \times \mathfrak{g}.$$

**Theorem (After Alekseev-Malkin-Meinrenken, Boalch, Hitchin, Jeffreys, Naef, Simpson, . . . )**

*The RH map is equivariant w.r.t to the action of the braid group of $C$, and if one formally identifies*

$$\mathrm{M}(C, P) \cong \left((T^*\mathfrak{g})^{2g} \times \mathfrak{g}^{n-1}\right)/G$$

*it is also Poisson.*

# Quantization

- The KZ equation has an analog (the KZB equation) for the configuration space of points on any Riemann surface: monodromy representation of the associated braid group on $(S(\mathfrak{g})^{\otimes g} \otimes V^{\otimes n})^{\mathfrak{g}}$ for any f.d. $\mathfrak{g}$-module $V$.
- Fact (Hamad, Reshetikhin, Takasaki, in particular cases): quantizes the braid action by isomonodromy.
- Calaque–Enriquez–Etingof: "universal" version in case $C$ is an elliptic curve: action of the braid group on $(M \otimes V^{\otimes n})^{\mathfrak{g}}$ where $M$ is any $D(\mathfrak{g})$-module.
- Recall that $D(\mathfrak{g})$ is a quantization of $T^*\mathfrak{g}$.

# Quantization

**Almost Theorem**

- *There exists a lift* $\tilde{D}(\mathfrak{g})$ *of* $D(\mathfrak{g})$ *to an algebra in* $\mathrm{Rep}_{\mathrm{KZ}}\, G$*, whose category of modules carries an action of the elliptic braid group which recovers the CEE one upon taking invariants/global sections.*
- *The equivalence* $\mathrm{Rep}_{\mathrm{KZ}}\, G \simeq \mathrm{Rep}_q\, G$ *extends to a formal equivalence*

$$\tilde{D}(\mathfrak{g})\text{-mod}_{\mathrm{Rep}_{\mathrm{KZ}}\, G} \simeq D_q(G)\text{-mod}_{\mathrm{Rep}_q\, G}$$

*intertwining the elliptic braid group actions on both sides.*

Thank you for your attention !