

Quantization of character varieties, topological field theories and the Riemann-Hilbert correspondence

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Rough plan

- Various point of views on $\text{Rep}_q G$
- Quantization of character varieties
- Quantum Riemann-Hilbert correspondence

Rep_q G from rep. theory

- Fix G a simple (or reductive) complex algebraic group and $q \in \mathbb{C}^\times$. For the purpose of this talk q is generic and I will often assume $G = \mathrm{SL}_2(\mathbb{C})$.
- Rep_q G: category of (integrable) modules over the quantum group associated with G . This is a q -deformation/quantization of Rep G .
- As mere categories Rep_q G \simeq Rep G , but it has a non-trivial and interesting braided monoidal (in fact ribbon) structure.
- Hence one gets representations of braids groups, and links invariants (e.g. the Kauffmann bracket/Jones polynomial).

$\text{Rep}_q G$ from skein theory

- In $\text{Rep}_q \text{SL}_2$, every map between tensor powers (of the same parity) of $V = \mathbb{C}^2$ is the image of some tangle (Schur–Weyl duality).
- In turn those are determined by the value of “cup” and “cap” (making V self-dual) by repeated applications of Kauffmann’s skein relations:

$$\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = q^{\frac{1}{2}} \left| \begin{array}{c} \text{---} \\ \text{---} \end{array} \right| \quad \left| \begin{array}{c} \text{---} \\ \text{---} \end{array} \right| + q^{-\frac{1}{2}} \begin{array}{c} \text{---} \\ \text{---} \end{array}$$


- Define the Temperley-Lieb category TL_q as having objects finite sequences of points on a disk, and morphisms (framed unoriented in that case) tangles modulo the above relations.

Theorem (Folklore, Tingley for the correct signs)

The assignment $\bullet \mapsto V$ realizes $\text{Rep}_q \text{SL}_2$ as the completion of TL_q under direct sums and splittings of idempotents.

$\text{Rep}_q G$ from CFT (aka the Drinfeld–Kontsevich integral)

- $\text{Rep}_{\text{KZ}} G = \text{Rep } G$ with the same tensor product.
- Braided tensor structure defined analytically by integrating the so-called Knizhnik–Zamolodchikov connection on the configuration space of points in \mathbb{C} .
- Transcendental and highly non-explicit, but geometric and conceptually clear.

Theorem (Kohno, Drinfeld, Kazhdan–Lusztig)

Equivalence of braided tensor categories

$$\text{Rep}_{\text{KZ}} G \simeq \text{Rep}_q G.$$

Rep_q G as a quantization

- Quasi-classical limit: set $q = 1 + \hbar$ and expand

$$\text{Diagram of a crossing} - \text{Diagram of two parallel lines} = \hbar t + O(\hbar^2)$$

where $t \in \mathfrak{g}^{\otimes 2}$ is the canonical element w.r.t the Killing form.

- (Rep G , t) is an “infinitesimal braided monoidal category”.
- Poisson geometric interpretation: the sym. mon. Rep G is naturally identified with quasi-coherent sheaves (= categorified algebra of functions) on the classifying stack $BG = pt/G$.
- t turns BG into a (2-shifted) symplectic stack in the sense of Pantev–Toën–Vaquié–Vezzosi.
- Rep_{KZ} G is the quantization of BG coming from the formality of the little disks operad.

Character varieties

- Let S be a topological surface. The **representation variety** of S is

$$R(S) := \{\pi_1(S) \rightarrow G\}$$

- The **character variety** of S is

$$CH(S) := R(S)/G^{ad}.$$

Theorem (Atiyah–Bott, Goldman, PTVV)

$CH(S)$ carries a canonical Poisson structure.

- Canonical \longrightarrow diffeomorphism invariant.
- Really depends on the surface, not just on its π_1 .

Character varieties

- Observe that $\mathrm{CH}(D^2) = pt/G = BG$.
- Hence QC of $\mathrm{CH}(S)$ obtained by gluing together copies of $\mathrm{Rep} G$.
- This is very non-true for the algebra of global functions (i.e. for the underlying variety as opposed to stack).

Theorem (Ben-Zvi–Francis–Nadler)

The assignment $S \mapsto \mathrm{QC}(\mathrm{CH}(S))$ is part of a 2d topological field theory.

- The Poisson structure is given by “integrating the Killing form”.
- Fairly simple recipe for its behaviour under gluing.

Slogan (Alekseev–Malkin–Meinrenken, Calaque, Safronov)

Gluing \longleftrightarrow (multiplicative) Hamiltonian reduction

Examples and building blocks

- Assume S has genus g and $n > 1$ boundary components.
- Then $\pi_1(S) = F_{2g+n-1}$ is free, hence one has a (non-canonical !) identification

$$R(S) \cong G^{2g+n-1}.$$

- It implies that

$$QC(\text{CH}(S)) \simeq \mathcal{O}(R(S))\text{-mod}_{\text{Rep } G}.$$

- Fock-Rosly: the Poisson structure on $\text{CH}(S)$ has a (again non-canonical) lift to $R(S)$.
- $R(\text{Annulus}) \cong G_{ST_S}$, Poisson structure discovered by Semenov-Tian-Shansky.
- $R(T^2 \setminus pt) \cong G \times G$ the "Heisenberg double", a multiplicative version of T^*G .
- Other punctured and closed surfaces obtained by combining those two and using Hamiltonian reduction.

Combinatorial quantization

- G_{STS} quantizes to $\mathcal{O}_q(G)$, a well-known algebra which goes by various names (braided dual, reflection equation algebra, ...). Discovered by Majid and Donin–Mudrov.
- The Heisenberg double quantizes to $D_q(G)$, a simultaneous q -deformation of $D(\mathfrak{g})$ and $D(G)$, which you might remember from Sam's talk this morning. This is $\mathcal{O}_q(G)^{\otimes 2}$ as an object, plus certain cross relations.
- Arbitrary punctured surfaces $S_{g,n}$: Alekseev's moduli algebras

$$D_q(G)^{\tilde{\otimes} \mathfrak{g}} \tilde{\otimes} \mathcal{O}_q(G)^{\tilde{\otimes} n-1}$$

(braided tensor product).

- Taking invariants leads to a quantization of $\mathcal{O}(\text{CH}(S))$.
- ✓ Explicit and algebraic
- ✗ Depends on a combinatorial presentation of S . Topological features have to be proved by hand.
- ✗ Originally not defined for closed surfaces.

Skein quantization

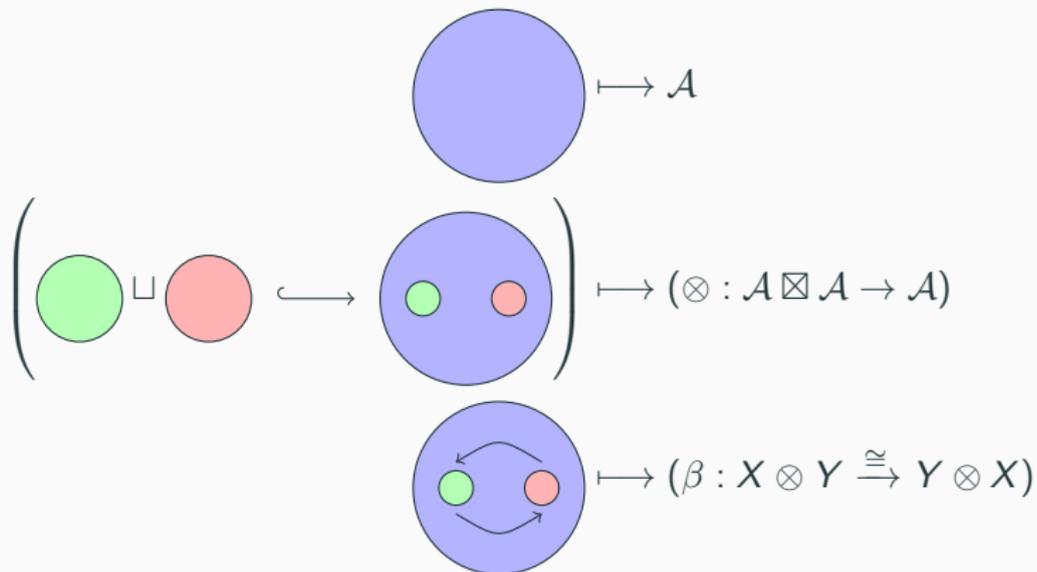
- If M is a 3-fold, the skein module $\text{Sk}(M)$ is the vector space of isotopy classes of links in M modulo the skein relation.
- Functorial w.r.t to embeddings. In particular, if S is a surface, $\text{Sk}(S) := \text{Sk}(S \times [0, 1])$ is an algebra.
- $\text{Sk}(S)$ is a quantization of $\text{CH}(S)$ (Bullock, Frohman, Przytycki, Turaev, ...)
- ✓ Elementary to define, manifestly topological.
- ✗ Difficult to handle algebraically (already for SL_2).
- ✗ No TFT/gluing formula around.

Factorization homology

- Want the best of both worlds and more: a purely topological, TFT like construction, together with an explicit combinatorial description.
- Idea: use factorization homology to quantize categories of sheaves on $\text{CH}(S)$.
- Let Mfld_2 be the category with:
 - objects oriented top. surfaces
 - morphisms smooth embeddings
 - 2-morphisms isotopies between those
 - (and so on although we won't need it).
- Sym. mon. structure given by disjoint union.
- Disk_2 is the full subcategory of finite disjoint unions of disks.

Factorization homology

- A ribbon category \mathcal{A} determines a sym. mon. functor $\text{Disk}_2 \rightarrow \text{Cat}_{\mathbb{C}}^{\boxtimes}$, a.k.a. an algebra over the little 2-disks/ E_2 operad



Theorem (Ayala–Francis–Tanaka)

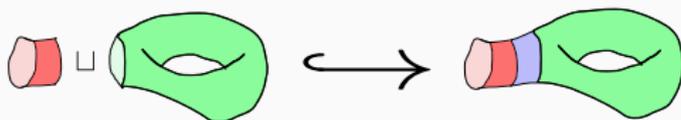
There exists a canonical extension

$$\begin{array}{ccc} \text{Mfld}_2 & & \\ \uparrow & \searrow \int_{(-)} \mathcal{A} & \\ \text{Disk}_2 & \xrightarrow{\mathcal{A}} & \text{Cat}_{\mathbb{C}}^{\boxtimes} \end{array}$$

*which satisfies and is characterized by **excision**.*

Factorization homology

- If $S = P \times [0, 1]$, by functoriality w.r.t embedding $\int_S \mathcal{A}$ is an algebra (i.e. a monoidal category).
- If S has boundary P , $\int_S \mathcal{A}$ is a module over it.



$$\int_{S^1 \times [0,1]} \mathcal{A} \boxtimes \int_{T^2 \setminus pt} \mathcal{A} \longrightarrow \int_{T^2 \setminus pt} \mathcal{A}$$

Excision

$$\int_{\substack{X \sqcup Y \\ P \times [0,1]}} \mathcal{A} \simeq \int_X \mathcal{A} \boxtimes \int_{P \times [0,1]} \mathcal{A} \int_Y \mathcal{A}$$

Main result

Theorem (Ben-Zvi–B–Jordan)

Any choice of a combinatorial decomposition of a punctured surface S induces an equivalence

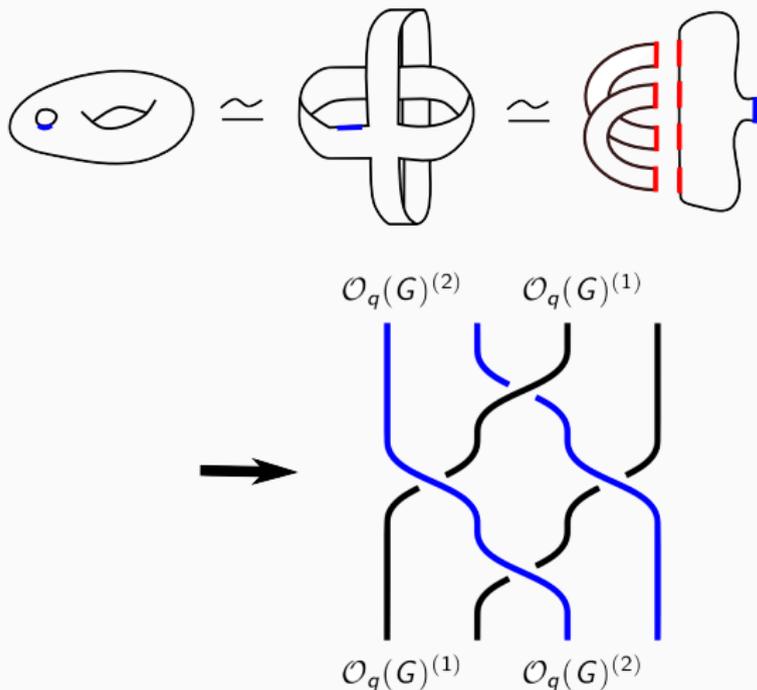
$$\int_{\text{Rep}_q G} S \simeq A_S\text{-mod}_{\text{Rep}_q G}$$

where A_S is the moduli algebra attached to the same decomposition.

- We get those algebras as an **output** of a topological machine.
- Similar albeit slightly more complicated picture for closed surfaces and surfaces with marked points.
- For the closed torus with a suitable marking we get a certain category of equivariant $D_q(G)$ -modules. . .
- . . . and upon taking global sections category of modules for the sDAHA (with both parameters and its $\text{SL}_2(\mathbb{Z})$ -action).

Braids and doughnuts are worth a thousand words

- One copy of $O_q(G)$ for each “flat handle”, cross-relations dictated by the way those are attached to the disk on the right.



Theorem (Cooke, Johnson-Freyd, after Walker)

$\int_{\text{Rep}_q G} S$ is identified with an appropriate completion of the *skein category* of S , having objects finite collections of points on S , and morphisms tangles in $S \times [0, 1]$ modulo the skein relation.

- Endomorphisms of the empty collection is the skein algebra. Hence, this identifies the skein and the combinatorial quantization of global functions on $\text{CH}(S)$.
- It shows that, in sharp contrast with skein algebras, skein categories satisfies excision.

3D extension

- If M is a 3-fold, $\text{CH}(M) \rightarrow \text{CH}(\partial M)$ is Lagrangian.
- Gluing \leftrightarrow Lagrangian intersection.
- With Jordan–Snyder we extend the previous construction to a 3d TFT (more on that on Thursday).
- This recovers skein modules and their relative versions.
- Of particular interest is the quantization of

$$\text{CH}(S^3 \setminus K) \longrightarrow \text{CH}(T^2)$$

for a knot K (quantum A-polynomial).

- Let C be a Riemann surface and P a collection of $n > 0$ points on it.
- Let $M(C, P)$ be the moduli space of meromorphic connections with regular singularities at each $x \in P$.
- The RH correspondence is an analytic isomorphism

$$M(C, P) \cong \text{CH}(S)$$

taking a connection to its monodromy representation, where S is the underlying surface of C with a disk removed around each $x \in P$.

- This induces an iso between the formal neighborhood of the trivial connection and the trivial representation respectively.
- The former can be identified with $\widehat{\mathfrak{g}}^{2g+n-1}/G$.

- The n th braid group of C acts on $M(C, P)$ by **isomonodromy**, i.e. by moving the marked points while keeping the monodromy constant, and on $\text{CH}(S)$ by moving the boundary components.
- Recall that \mathfrak{g}^* has a linear Poisson structure coming from the Lie bracket.
- The Killing form gives G -equivariant identifications

$$\mathfrak{g} \cong \mathfrak{g}^* \qquad T^*\mathfrak{g} \cong \mathfrak{g} \times \mathfrak{g}.$$

Theorem (After Alekseev-Malkin-Meinrenken, Boalch, Hitchin, Jeffreys, Naef, Simpson, ...)

The RH map is equivariant w.r.t to the action of the braid group of C , and if one formally identifies

$$M(C, P) \cong ((T^*\mathfrak{g})^{2g} \times \mathfrak{g}^{n-1}) / G$$

it is also Poisson.

Quantization

- The KZ equation has an analog (the KZB equation) for the configuration space of points on any Riemann surface: monodromy representation of the associated braid group on $(S(\mathfrak{g})^{\otimes g} \otimes V^{\otimes n})^{\mathfrak{g}}$ for any f.d. \mathfrak{g} -module V .
- Fact (Hamad, Reshetikhin, Takasaki, in particular cases): quantizes the braid action by isomonodromy.
- Calaque–Enriquez–Etingof: “universal” version in case C is an elliptic curve: action of the braid group on $(M \otimes V^{\otimes n})^{\mathfrak{g}}$ where M is any $D(\mathfrak{g})$ -module.
- Recall that $D(\mathfrak{g})$ is a quantization of $T^*\mathfrak{g}$.

Almost Theorem

- *There exists a lift $\tilde{D}(\mathfrak{g})$ of $D(\mathfrak{g})$ to an algebra in $\text{Rep}_{KZ} G$, whose category of modules carries an action of the elliptic braid group which recovers the CEE one upon taking invariants/global sections.*
- *The equivalence $\text{Rep}_{KZ} G \simeq \text{Rep}_q G$ extends to a formal equivalence*

$$\tilde{D}(\mathfrak{g})\text{-mod}_{\text{Rep}_{KZ} G} \simeq D_q(G)\text{-mod}_{\text{Rep}_q G}$$

intertwining the elliptic braid group actions on both sides.

Thank you for your attention !