Dualizable and invertible braided tensor categories

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11 May 2020

Main motivation

- There are natural notions of dualizability and invertibility in (possibly higher) symmetric monoidal categories.
- On the one hand, these recover many important notions in geometry and rep. theory.
- On the other hand, there is a remarkable connection with topology via topological field theories and Baez–Dolan–Lurie's cobordism hypothesis.
- Original motivation: topological invariants which are easy to compute algebraically.
- Can also be used to provide topological interpretations/proofs of algebraic properties, structures or statements.
- Ultimately I'd like to say some things about so-called non-degenerate finite braided tensor categories in that framework.

Plan

1. Recollection on TFT's and the cobordism hypothesis
2. Dualizability of finite tensor categories
3. Dualizability of braided tensor categories
4. Invertibility of finite braided tensor categories

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Invertible and dualizable objects

- Recall that an object X in a symmetric monoidal category $\mathcal{S}$ is *dualizable* if there exists some object X^* and maps

$$\mathrm{ev} : X^* \otimes X \to \mathbf{1}_{\mathcal{S}} \qquad \mathrm{coev} : \mathbf{1}_{\mathcal{S}} \to X \otimes X^*$$

  satisfying the usual zig-zag equation.
- X is *invertible* if either one of ev or coev is an isomorphism.
- This makes sense in higher (e.g. (∞, n)) categories by requiring those equations to hold up to equivalence.
- The Picard groupoid $\mathrm{Pic}(\mathcal{S})$ has objects invertible objects in $\mathcal{S}$, and morphisms isomorphisms between those. Its set of isomorphism classes of objects has a natural group structure, the Picard group.
- The classical example is: in the category of quasi-coherent sheaves over a scheme S, dualizable objects are vector bundles and invertible objects are line bundles.

TFT's: first version

Definition (Atiyah)

A topological field theory is a symmetric monoidal functor

$$Z : \widetilde{\mathrm{Bord}}_n \longrightarrow \mathcal{S}$$

where $\widetilde{\mathrm{Bord}}_n$ *has objects closed oriented* $n-1$*-dim topological manifolds, morphisms n-dimensional bordisms between these and tensor product given by disjoint union. A TFT is invertible if it factors through* $\mathrm{Pic}(\mathcal{S})$.

- Z maps a closed n-manifold to a map $\mathbf{1}_{\mathcal{S}} \to \mathbf{1}_{\mathcal{S}}$, e.g. a number if $\mathcal{S} = \mathrm{Vect}_{\mathbf{k}}$ for some field **k**.
- A crucial point is that an n-manifold with boundary $X \sqcup \bar{Y}$ ($\bar{Y}$ is Y with opposite orientation) can be seen as a morphism in this category in many different ways. This already imposes strong finiteness conditions, in particular that $Z(X)$ has to be dualizable.

Kitten version of the cobordism hypothesis

In dimension 1 there isn't much more you can do:

Theorem

$\mathcal{S}$-valued 1d TFT's (resp. invertible TFT's) are in one to one correspondence with dualizable (resp. invertible) objects in $\mathcal{S}$. The image of the circle is

$$\dim(X) : \mathbf{1}_{\mathcal{S}} \longrightarrow \mathbf{1}_{\mathcal{S}}.$$

TFT's in low dimension

- (Folklore) oriented 2d TFT's are classified by Frobenius algebra in $\mathcal{S}$ (i.e. dualizable objects equipped with a commutative algebra structure and a non-degenerate invariant pairing).
- Reshetikhin–Turaev and Turaev–Viro: two closely related constructions of a 3d TFT from a modular and a spherical fusion category respectively.
- To get a general classification, one needs to consider TFT's that attach objects to lower-dimensional manifolds, down to the point.
- The TV TFT is an example, but neither the Frobenius algebra nor the RT one are. However, the RT TFT can be recovered from an invertible 4D TFT which fits into that framework and is one of the motivations for this talk.

The bordism category

Definition–Theorem (Lurie, Calaque-Scheimbauer)

There exists a symmetric monoidal (∞, n) *category* Bord_n *which has*

- *objects: disjoint unions of framed points*
- *k-morphisms for $k \leq n$: k-dimensional framed manifolds with corners*
- *$n+1$-morphisms: diffeomorphisms between framed n-manifolds with corners*
- *$n+2$-morphisms: isotopies between those*
- *etc...*

The cobordism hypothesis

Definition

A framed fully extended (or fully local) n-dimensional TFT is a functor of symmetric monoidal ∞-categories

$$Z : \mathrm{Bord}_n \longrightarrow \mathcal{S}.$$

It is invertible if it factors through the Picard ∞-groupoid $\mathrm{Pic}(\mathcal{S})$.

Definition

An object in $\mathcal{S}$ is called

- *1-dualizable if it's dualizable in the usual sense*
- *2-dualizable if* ev *and* coev *have a left and a right adjoint, if these adjoints have adjoints and so on*
- *3-dualizable if the structure maps realizing these adjunctions, themselves have adjoints on both sides, etc...*
- *...*

The cobordism hypothesis

Theorem (Lurie, Ayala–Francis)

Evaluation on the point gives an equivalence between $\mathcal{S}$-valued framed fully extended n-dimensional TFT's (resp. invertible TFT's) and n-dualizable (resp. invertible) objects in $\mathcal{S}$.

Remark

An invertible object is automatically k-dualizable for all k.

Example from Morita theory

- Let R be a commutative ring, and let $\mathrm{Alg}_1(R)$ be the symmetric monoidal 2-category defined as follows
  - objects are associative R-algebras, and the tensor product is $\otimes_R$.
  - morphisms $A \to B$ are $A - B$-bimodules, composition is relative tensor product
  - 2-morphisms are morphisms of bimodules
- We have the following
  - Every object A in $\mathrm{Alg}_1(R)$ is dualizable, with dual given by the opposite algebra A^{op}.
  - The dimension of A is its cocenter $A/[A, A]$, a.k.a. its 0th Hochschild homology.
  - 2-dualizable objects in $\mathrm{Alg}_1(R)$ are algebras which are finitely generated projective over R, and separable.
  - Invertible objects are central simple (a.k.a Azumaya) algebras: A is invertible iff it is finitely generated and projective as an R-module, and there is an algebra isomorphism $A \otimes_R A^{op} \cong \mathrm{Hom}_R(A, A)$
  - There is an oriented version: *separable, non-commtative* Frobenius algebras, and value on the circle is its center.

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Finite tensor categories

From now on $\mathbf{k}$ is a field of char 0.

Definition

A $\mathbf{k}$-category $\mathcal{C}$ is finite if it is equivalent to A-mod for some finite dimensional $\mathbf{k}$-algebra A.

Theorem (Deligne)

The $(2,1)$-category $\mathrm{FinCat}_{\mathbf{k}}$ of finite categories and right exact functors is symmetric monoidal, with tensor product $\boxtimes$ defined in such a way that

$$A\text{-mod} \boxtimes B\text{-mod} \simeq (A \otimes B)\text{-mod}\,.$$

Finite tensor categories

Definition

A finite tensor category is an algebra in $(\mathrm{FinCat}_{\mathbf{k}}, \boxtimes)$ *which is rigid, i.e. a finite, rigid monoidal category* $\mathcal{A}$ *such that the tensor product is right exact in each variable, equivalently such that the tensor product factors through*

$$\mathcal{A} \times \mathcal{A} \to \mathcal{A} \boxtimes \mathcal{A} \to \mathcal{A}.$$

Definition

A semi-simple finite tensor category is called a fusion category.

Remark

Finite tensor categories generalize the rep. theory of f.d. Hopf algebras. Fusion categories, in particular, generalize the rep. theory of finite groups.

Morita theory of tensor categories

- (Lurie, Haugseng, Scheimbauer, Johnson-Freyd–Scheimbauer) The construction of the Morita category makes sense for algebras in any symmetric monoidal $(\infty, 1)$-category $\mathcal{S}$ satisfying some assumptions. Again, every algebra A in $\mathrm{Alg}_1(\mathcal{S})$ is automatically 1-dualizable, with dual A^{op}.
- There is a technical caveat that $\mathrm{FinCat}_{\mathbf{k}}$ does not quite satisfy these assumptions, so strictly speaking one has to work in the larger world of locally presentable categories $\mathrm{LP}_{\mathbf{k}}$.
- Hence this produces a symmetric monoidal 3-category whose objects are monoidal categories in $\mathrm{LP}_{\mathbf{k}}$, morphisms bimodules over those and 2-morphisms colimit preserving bimodule functors.

Theorem (Douglas–Schommer-Pries–Snyder)

- *Every finite tensor category is 2-dualizable in* $\mathrm{Alg}_1(\mathrm{LP}_{\mathbf{k}})$.
- *A finite tensor category is 3-dualizable iff it is fusion.*

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Morita theory of braided tensor categories

- Recall that if $\mathcal{A}$ is a monoidal category, its Drinfeld center $Z(\mathcal{A})$ is a braided monoidal category which has objects $(x, \beta_{x,-})$ where $x \in \mathcal{A}$ and

$$\beta_{x,-} : x \otimes - \xrightarrow{\simeq} - \otimes x$$

  is a natural isomorphism satisfying the axioms for a braiding. This is the categorification of the commutative center of an associative algebra.

- if $\mathcal{A}$ is a braided monoidal category, a monoidal category over $\mathcal{A}$ is a monoidal category $\mathcal{B}$ together with a braided monoidal functor

$$\mathcal{A} \longrightarrow Z(\mathcal{B}).$$

  This is the categorification of the notion of algebra over a commutative algebra.

Morita theory of braided tensor categories

- The construction of Haugseng and Scheimbauer produces a Morita category of braided monoidal categories, as a particular case of a general construction for E_n-algebras, a.k.a. algebras over the little n-disks operad.
- Results of Gwilliam–Scheimbauer guarantees braided monoidal categories are at least 2-dualizable in this Morita category.
- The same technical caveat as for tensor categories apply.

Theorem (Haugseng, Scheimbauer, Johnson-Freyd–Scheimbauer, Scheimbauer–Gwilliam)

There exists a symmetric monoidal 4-category $\mathrm{Alg}_2(\mathrm{LP}_{\mathbf{k}})$ *whose*

- *objects are braided monoidal categories in* $\mathrm{LP}_{\mathbf{k}}$
- *morphisms* $\mathcal{A} \to \mathcal{B}$ *are monoidal categories over* $\mathcal{A} \boxtimes \mathcal{B}^{op}$.
- *2-morphisms are bimodules over those*
- *3-morphisms are colimit preserving bimodule functors.*

Dualizability of braided tensor categories

Theorem (B–Jordan–Snyder)

- *Finite braided tensor categories are 3-dualizable in* $\mathsf{Alg}_2(\mathsf{LP}_{\mathbf{k}})$.
- *Braided fusion categories are 4-dualizable.*

- 3-dualizability holds more generally for rigid braided tensor categories with enough projectives.
- Fusion is only sufficient, not necessary for 4-dualizability, in sharp contrast with what happens for tensor categories.
- However, in the fusion case the technical caveats disappear and there is a Morita category $\mathsf{BrFus}_{\mathbf{k}} \subset \mathsf{Alg}_2(\mathsf{LP}_{\mathbf{k}})$ where everything in sight is finite and rigid.

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Main result

- Recall that the Müger center of a braided tensor category $\mathcal{A}$ is the full sub-category of transparent objects, i.e. those $x \in \mathcal{A}$ such that

$$\forall y \in \mathcal{A},\ \beta_{x,y}\beta_{y,x} = \mathsf{Id}_{x,y}\,.$$

- A finite braided tensor category is called *non-degenerate* if its Müger center is trivial.
- Hence non-degenerate braided categories are "maximally non-symmetric". This generalizes the notion of factorizable Hopf algebras.

Theorem (B–Jordan–Safronov–Snyder)

A finite braided tensor category is invertible in $\mathrm{BrTens}_{\mathbf{k}}$ *iff it is non-degenerate.*

Comments

- We do not know if invertibility forces finiteness.
- The cobordism hypothesis then implies that every non-degenerate finite braided tensor category gives rise to a fully extended invertible 4d TFT.
- If $\mathcal{A}$ is pivotal (i.e. if there is a natural monoidal iso $(-) \simeq (-)^{**}$) and semi-simple, then it is non-degenerate iff it is modular. In that case, this result has been anticipated for many years, and proved in unpublished work of Freed-Teleman and Walker. This relies on a construction of the RT TFT from the cobordism hypothesis.
- However, it was widely expected than any TFT going all the way up (i.e. producing numbers in top dimension) would require some sort of semi-simplicity condition, so our result is somewhat surprising in that respect.
- Interesting non-semi-simple examples come from rep. theory of small quantum groups at roots of unity.

Equivalent characterizations

- It was shown by Müger and Etingof–Gelaki–Nikshych–Ostrik in the semi-simple case, and by Shimizu in the general case, that a finite braided tensor category is non-degenerate iff it is factorizable, meaning the canonical braided functor

$$\mathcal{A} \boxtimes \mathcal{A}^{op} \longrightarrow Z(\mathcal{A})$$

  is an equivalence.
- We show this is in turn equivalent to $\mathcal{A}$ being co-factorizable, meaning that a certain monoidal functor

$$HC(A) \longrightarrow \mathrm{Hom}(\mathcal{A}, \mathcal{A})$$

  is an equivalence, where $HC(A) = \mathcal{A} \boxtimes_{\mathcal{A} \boxtimes \mathcal{A}^{op}} \mathcal{A}^{op}$ is what we call the Harish-Chandra category of $\mathcal{A}$.
- We then show a rigid but not *à priori* finite braided tensor category is invertible iff it is factorizable, co-factorizable and non-degenerate. In fact, we prove a similar statement for general E_2-algebras.

The Witt group

- One of the motivations for non-degeneracy is to obtain results in the classification of finite braided tensor categories.
- The Drinfeld center of a finite tensor category $\mathcal{A}$ is always finite and non-degenerate, and it is fusion whenever $\mathcal{A}$ is.
- Davydov–Müger–Nikshych–Ostrik introduced the WItt group of braided fusion categories, which roughly speaking is the group of equivalence classes of braided fusion categories modulo Drinfeld centers.
- More precisely, two braided fusion categories $\mathcal{A}, \mathcal{B}$ are called Witt-equivalent if there exist fusion categories $\mathcal{C}$ and $\mathcal{D}$ and a braided equivalence

$$\mathcal{A} \boxtimes Z(\mathcal{C}) \simeq \mathcal{B} \boxtimes Z(\mathcal{D}).$$

- The set of equivalence classes of non-degenerate braided fusion categories is a group, with inverse given by $[\mathcal{A}]^{-1} = [\mathcal{A}^{op}]$, due to the factorizability property.

The Witt group

- We prove the following:

Theorem (B–Jordan–Safronov–Snyder)

The Witt group is isomorphic to the Picard group of the Morita category $\mathrm{BrFus}_{\mathbf{k}}$ *of braided fusion categories.*

- Roughly speaking, this means Drinfeld centers of fusion categories are not just invertible, but trivial in $\mathrm{BrFus}_{\mathbf{k}}$ (this is a reformulation of a result by Jones–Morrison–Penneys–Plavnik, and is also to be expected from the TFT perspective). We do not know if this is true without the fusion condition.
- We get this way a morphism from the Witt group, to the Picard group of the Morita category $\mathrm{BrTens}_{\mathbf{k}}$ of braided tensor category.

Some open questions

- Is this morphism injective ?
- Is it surjective ? In other words, is an invertible braided tensor category necessarily Morita equivalent to a braided fusion one ?
- Is an invertible braided tensor category necessarily finite ? We know it has to be close to finite.
- Are there examples of 4-dualizable braided tensor categories which are neither invertible nor fusion ? We believe this might be the case for finite braided tensor categories whose Müger center is semi-simple.

Thanks for your attention !

Bonus: RT TFT after Freed-Teleman and Walker

- $\mathcal{A}$ non-degenerate finite braided tensor category $\to$ invertible framed 4D TFT $Z_{\mathcal{A}}$.
- $\mathcal{A}$ pivotal/ribbon, conjecturally $Z_{\mathcal{A}}$ descends to an oriented TFT.
- Up to now this depends on $\mathcal{A}$ only up to Morita equivalence.
- $\mathcal{A}$ moreoever fusion (i.e. modular) $\to$ morphism of TFT from the trivial one to $Z_{\mathcal{A}}$. This now depends on $\mathcal{A}$ up to braided equivalence.
- Hence if X is a closed 3-fold, $Z_{\mathcal{A}}(X)$ is 1-dimensional and we get a map $Z_{\mathcal{A}}(X) \to \mathbf{k}$.
- If W is such that $\partial W = X$, get a map $\mathbf{k} \xrightarrow{Z_{\mathcal{A}}(W)} Z_{\mathcal{A}}(X)$ which depends on W up to bordism.
- The RT invariant of X is the composition

$$\mathbf{k} \longrightarrow Z_{\mathcal{A}}(X) \longrightarrow \mathbf{k}\,.$$

- If now $\mathcal{A}$ is the center of a spherical fusion category $\mathcal{C}$, $Z_{\mathcal{A}}$ is trivial and we get an honest fully 3D TFT, TV of $\mathcal{C}$.